A Simplified Stabilizer zx-calculus

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The zx-calculus is a high-level and intuitive graphical language for pure qubit quantum mechanics (QM), based on category theory [4]. It comes with a set of rewrite rules that potentially allow this graphical calculus to be used to replace matrix-based formalisms entirely for certain classes of problems. However, this replacement is only possible without losing deductive power if the zx-calculus is *complete* for this class of problems, i.e. if any equality that is derivable using matrices can also be derived graphically.

The overall zx-calculus for pure state qubit quantum mechanics is incomplete [11, 9], and it is not obvious how to complete it in general [11]. Yet, the zx-calculus is complete for stabilizer quantum mechanics [6]. Stabilizer QM is a restricted fragment of quantum theory – in fact, it is efficiently classically stimulable [7] – that nevertheless exhibits many important quantum properties, like entanglement and non-locality. It is furthermore of central importance in areas such as quantum error correcting codes [8] and measurement-based quantum computation [10].

The completeness of the stabilizer zx-calculus has been established first in a setting where scalars (i.e. diagrams with no input nor output) are ignored [1]. In this setting, when two diagrams are equal according to the rules of the language, their matrices are equal up to a non-zero scalar factor. To expand the completeness result to the stabilizer zx-calculus with scalars, a new symbol and three rules were then added to the original zx-calculus [2].

While the stabilizer zx-calculus is thus known to be complete, little is known about which rules are actually needed for completeness. The Euler decomposition of the Hadamard node – here given as (EU) in Figure 3 – was shown to be independent of the other then-existing rewrite rules upon its introduction [5]. Yet so far there are no arguments concerning the necessity of any of the original rules in [4], nor for the new rules introduced in [2].

In our paper, we simplify the proof of completeness for the scaled stabilizer zx-calculus by showing that there is no need to introduce a new symbol for making the language complete for scalars. Moreover, we show that one of the three new axioms is not necessary and can be derived from the rest of the language. We end up with only two rules for scalars, both of which are proved to be necessary.

Beyond the treatment of scalars, we also consider how to simplify the set of rules for the full stabilizer zx-calculus. Usually, in addition to about a dozen explicitly-stated rewrite rules, there is a convention that any rule also holds with the colours red and green swapped or with the diagrams flipped upside-down, effectively nearly quadrupling the available set of rewrite rules[1]. Furthermore, the zx-calculus generally includes a meta-rule 'only the topology matters', which means that two diagrams represent the same matrix whenever one can be transformed into the other by moving components around without changing their connections.

We replace the topology meta rule with an explicit set of rewrite rules based on the properties of a compact closed category: we assume the existence of a swap map (denoted by a wire crossing) and

[1]Some rules are symmetric under the operations of swapping the colours and/or flipping them upside-down, hence the effective rule set is not quite four times the size of the explicitly-given one.

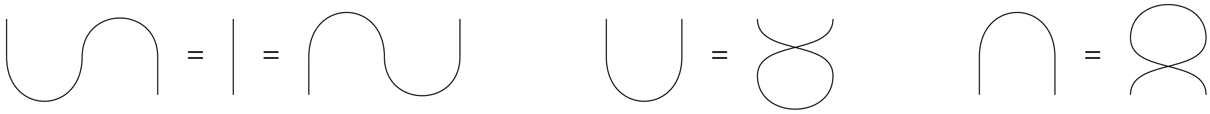


Figure 1: Rewrite rules of a compact closed category: snake equations and symmetry of cups and caps.

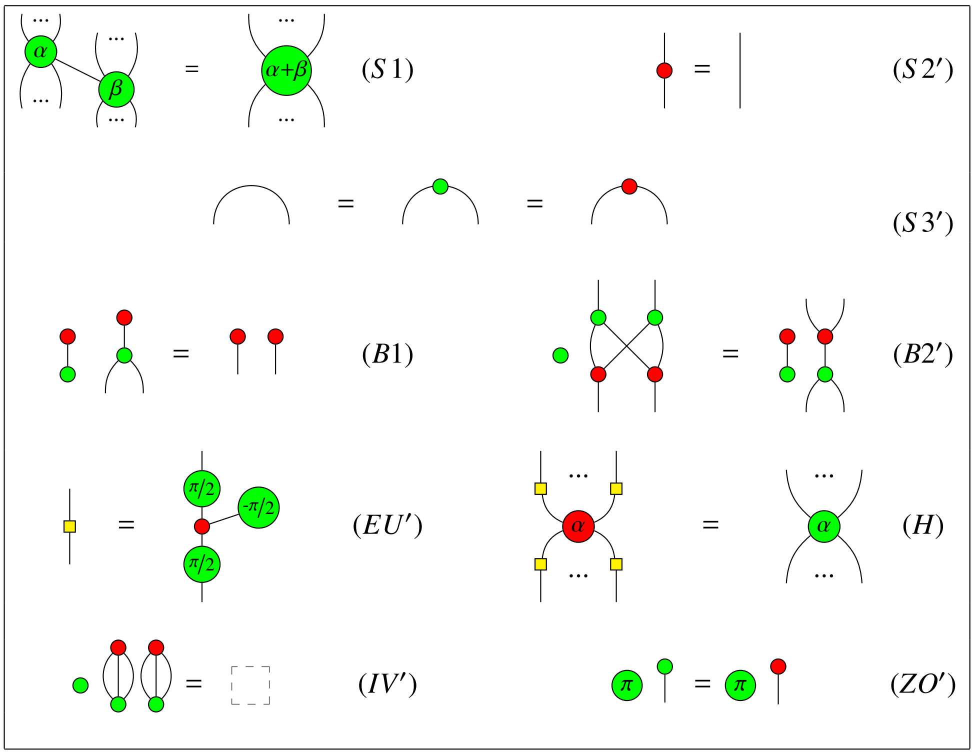
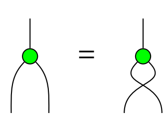


Figure 2: The simplified rules for the stabilizer zx-calculus with scalars. The right-hand side of (IV′) is an empty diagram and ellipses denote zero or more wires.

'cups' and 'caps' satisfying the equations in Figure 1, as well as the fact that arbitrary maps can slide freely along either wire of a swap.

We then give a new system of just nine rules, shown in Figure 2, for the stabilizer zx-calculus with scalars. We prove that all of these rules are sound by showing that they can be derived from the old rules. Furthermore, we prove that all the old rules, including their colour-swapped and upside-down versions, can be derived from the new system of rules together with the axioms of a compact closed category. We also derive the topology meta rule, which – under the assumption of the axioms of a compact closed category – reduces to proving the commutativity of the green copy map:



and its colour-swapped and upside-down versions.

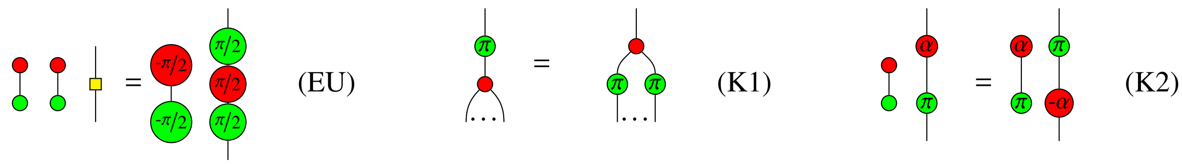


Figure 3: The Euler decomposition rule (EU), the π-copy rule (K1) and the π-commutation rule (K2).

Our results show that the π-copy rule, labelled (K1) in Figure 3, one of the original axioms of the zx-calculus, is not actually necessary: it can be derived from the other rules. We furthermore show that – within stabilizer quantum mechanics, i.e. for angles that are integer multiples of $\frac{\pi}{2}$ – the π-commutation rule, labelled (K2) in Figure 3, is also derivable. Our version of the spider law, labelled (S1) in Figure 2, has exactly one edge connecting the two spiders; nevertheless self-loops on spiders such as might result from merging two spiders with more connecting legs can be removed without needing an explicit 'loop rule' as assumed e.g. in [2]. We additionally eliminate almost all the colour-swapped or upside-down duplicates of rewrite rules: the only rule that still comes in two versions is (S3′).

As our rewrite rules imply the usual set, which is known to be complete, our rule set is complete too. In other words, we have a greatly simplified stabilizer zx-calculus. This work is an important step towards finding a minimal complete rule set for the stabilizer zx-calculus, i.e. a complete rule set in which each rule is provably independent of the others.

The full paper can be found at arXiv:1602.04744 [3].

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