Cohomology of effect algebras

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We will define cohomology groups of effect algebras, which occur in the algebraic study of quantum logic. Our definition is based on Connes' cyclic cohomology. The resulting cohomology groups are related to the state space of the effect algebra, and can be computed using variations on the Künneth and Mayer–Vietoris sequences. Using a slightly different version of the definition, we obtain applications to no-go theorems in quantum foundations, such as Bell's theorem.

1 Introduction

Cohomology groups can be assigned to various mathematical structures, such as topological spaces or groups, and are useful for classifying certain properties of the structures. In particular, they provide information about whether it is possible to pass from local information to global information. The differences between classical and quantum physics also involve this connection between local and global information [1], which suggests that cohomology theory may yield new insights into these differences.

Effect algebras form an abstract framework for unsharp measurements in quantum mechanics. Our goal is to define cohomology of effect algebras, and to study its applications to no-go theorems in quantum foundations. This note summarizes [7].

2 Effect algebras

Quantum mechanics features many counterintuitive phenomena. Therefore several techniques that are used to study systems in classical mechanics do not apply to quantum mechanical systems. We will take a look at the logical structures necessary for modeling quantum systems. Boolean algebras form a familiar algebraic framework for classical logic, but they are not suitable for quantum logic, since the distributive law is violated in quantum mechanics. There are several proposals for non-distributive generalizations of Boolean algebras. We will employ effect algebras, introduced in [4].

Definition 1. An *effect algebra* is a set A equipped with a partial binary operation $\boxplus$, constants $0,1 \in A$, and an orthocomplementation $(-)^{\perp} : A \to A$, such that:

- The partial operation $\boxplus$ is commutative, associative, and has 0 as neutral element.
- For every $a \in A$, the complement $a^{\perp}$ is the unique element for which $a \boxplus a^{\perp} = 1$.
- $0^{\perp} = 1$.
- If $a \boxplus 1$ is defined, then $a = 0$.

Basic examples include the unit interval $[0,1] \subset \mathbb{R}$, any orthomodular poset, and the collection of effects on a Hilbert space.

Effect algebras are helpful for investigating several problems in the foundations of quantum mechanics. For example, they are used to connect states and observables [5], and to determine which quantum mechanical procedures can also be realized classically [8].

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3 Cohomology

Since several facts from quantum foundations can be phrased in terms of effect algebras, we will define cohomology groups of effect algebras. This enables us to apply tools from homological algebra to obtain information about effect algebras. It turns out that a variation of Connes' cyclic cohomology [3] is the most natural way to assign cohomology groups to an effect algebra. These groups are constructed from the sets of *tests* on the algebra. A test on an effect algebra A is a sequence of elements $a_0, a_1, \ldots, a_n$ for which $a_0 \boxplus a_1 \boxplus \cdots \boxplus a_n$ is defined and equals 1. Roughly speaking, the construction of the cohomology groups proceeds as follows. Denote the collection of $(n+1)$-tests on A by $T_n(A)$. Then the collection of all tests can be made into a cochain complex if we define

$$\mathscr{C}^n(A) = \{\varphi : T_n(A) \to \mathbb{R} \mid \varphi(a_0, a_1, \ldots, a_n) = (-1)^n \varphi(a_1, a_2, \ldots, a_n, a_0)\}.$$

The coboundary maps are given by an alternating sum over face maps. The cyclic cohomology of A is then given by the cohomology groups $H^0(A)$, $H^1(A)$, $H^2(A)$, ... of the resulting cochain complex. The crucial step in this construction is restricting to the maps that are invariant under cyclic permutations (up to a sign), since this will make sure that the cohomology theory satisfies many interesting properties.

4 Properties

Cohomology detects various properties of the effect algebra associated to a physical system. For example, the *state space* of an effect algebra A is the space of maps from A to the unit interval $[0,1]$ that preserve partial addition and orthocomplements. Physically, the state space represents probability measures on the system under consideration. The state space can always be embedded into an $\mathbb{R}$-vector space; in fact, there is a smallest vector space in which it embeds. The cohomology of A contains information about the state space in the following way.

Theorem 2. *The first cohomology group $H^1(A)$ (with coefficients in $\mathbb{R}$) is the smallest vector space in which the state space of A can be embedded.*

Another property of cyclic cohomology is that it interacts well with products and unions of effect algebras. To prove this, it is essential that we use *cyclic* cohomology; the analogue for other cohomology theories is false in the case of effect algebras. The Künneth and Mayer–Vietoris sequences from homological algebra provide precise formulations of these assertions.

Theorem 3 (Künneth sequence)**.** *Let A and B be effect algebras. There is a long exact sequence*

$$\cdots \longrightarrow H^{n-1}(A\times B) \longrightarrow \bigoplus_{p+q=n-2} H^p(A)\otimes H^q(B) \longrightarrow \bigoplus_{p+q=n} H^p(A)\otimes H^q(B) \longrightarrow H^n(A\times B) \longrightarrow \cdots$$

Theorem 4 (Mayer–Vietoris sequence)**.** *If A and B are subalgebras of an effect algebra E, then there is a long exact sequence*

$$\cdots \longrightarrow H^{n-1}(A\cap B) \longrightarrow H^n(A\cup B) \longrightarrow H^n(A)\oplus H^n(B) \longrightarrow H^n(A\cap B) \longrightarrow H^{n+1}(A\cup B) \longrightarrow \cdots$$

These two results together allow calculation of the cohomology groups of all finite orthomodular posets, considered as effect algebras. The idea behind this is that the cohomology of all finite Boolean algebras can be computed using the Künneth sequence. Since each orthomodular poset is the union of its Boolean subalgebras, the Mayer–Vietoris sequence yields a technique to obtain its cohomology. This fits in a common theme in physics: if a quantum system is too complicated to study directly, it can sometimes be studied by first studying its classical subsystems and gluing these together. Classical subsystems are represented by Boolean subalgebras in our framework, and the Mayer–Vietoris sequence implements the gluing procedure.

5 Applications

Cohomology provides a method to check which quantum mechanical scenarios can also be realized classically. This was already observed in [2], and the effect algebraic account is similar. Represent the scenario by a state $\sigma : A \to [0,1]$ on an effect algebra A. Since Boolean algebras model classical systems, we say that the state σ is classically realizable if it factors through a Boolean algebra, i.e. there is a decomposition $\sigma = (A \xrightarrow{i} B \xrightarrow{\tau} [0,1])$, where B is a Boolean algebra.

To obtain a cohomological characterization for classical realizability, we need a variation on the cyclic cohomology defined above that also takes the order structure on effect algebras into account. This leads to the notion of *order cohomology* of effect algebras, generalizing a loosely related notion from [6]. Using this construction, one can show that a state $\sigma : A \to [0,1]$ is classically realizable if and only if a certain class in the second order cohomology group associated to σ is zero. For example, the Bell scenario can be modeled by a certain state on a suitable effect algebra [8]; since this scenario cannot be realized classically, the associated cohomology class is non-zero. An advantage of the effect algebraic approach above [2] is that we get an equivalent condition for classical realizability. In other words, no false positives will occur when using order cohomology of effect algebras.

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