Noise and Disturbance of Qubit Measurements: An Information-Theoretic Characterisation

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Information-theoretic definitions for the noise associated with a quantum measurement and the corresponding disturbance to the state of the system have recently been introduced [4]. These definitions are invariant under relabelling of measurement outcomes, and lend themselves readily to the formulation of state-independent uncertainty relations both for the joint-estimate of observables (noise-noise relations) and the noise-disturbance tradeoff. In this contribution we derive such relations for incompatible qubit observables, which we prove to be tight in the case of joint-estimates, and present progress towards fully characterising the noise-disturbance tradeoff. In doing so, we show that the set of obtainable noise-noise values for such observables is convex, whereas the conjectured form for the set of obtainable noise-disturbance values is not. Furthermore, projective measurements are not optimal with respect to the noise-disturbance or joint-measurement noise tradeoffs. Interestingly, it seems that three-outcome measurements are optimal in the former case, whereas four-outcomes are needed in the latter.

1 Introduction

Heisenberg's uncertainty principle is one of the defining nonclassical features of quantum mechanics, and expresses one of the fundamental physical consequences of the noncommutativity of quantum observables. Informally, the principle states that the measurement of one quantum observable (such as the position of a particle, x) introduces an irreversible disturbance into any complementary observable of the system (such as the particle's momentum, p), thus rendering it impossible to simultaneously measure, with arbitrary precision, the values of incompatible observable quantities.

Heisenberg's original presentation of the uncertainty principle, exhibited in his microscope *Gedankenexperiment* [10], was rather informal, and despite the evident physical importance of the principle it was a long time before it was rigorously formalised. Instead, subsequent theoretical work on the incompatibility of quantum observables focused on the inability to produce states with sharply defined values associated with noncommuting observables. These results are typically expressed in the form of uncertainty relations for the standard deviations of such observables – such as Kennard's well known relation [11] $\Delta x \Delta p \geq \frac{\hbar}{2}$ – and express a subtly different, although related, physical consequence of noncommutativity. To avoid confusion, we will call such relations *preparation uncertainty relations*.

It is only much more recently that, with the help of a more modern theory of quantum measurement [12], more rigorous formalisations of noise and disturbance operators, e.g. based on the root-mean-square distance between target observables and the measurement made [14], have become possible. This has allowed Heisenberg's uncertainty principle to be formalised in terms of *measurement uncertainty relations* between such noise and disturbance operators, although there still remains debate as to the most appropriate measures of noise and disturbance [3, 5, 8, 9, 14]. In fact, one may distinguish further two forms of measurement uncertainty relations expressing the incompatibility of such measurements [5]:

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noise-disturbance relations, expressing the tradeoff between the precision of a measurement and the subsequent disturbance to the state with respect to a complementary observable; and *noise-noise relations* for joint measurements, expressing the tradeoff in precision with which two complementary observables can be simultaneously measured.

Perhaps motivated by the success of entropic (preparation) uncertainty relations [6], which use entropy rather than the standard deviation to measure the uncertainty associated with an observable for a given state, a recent proposal set out a new approach to quantifying the noise and disturbance associated with a measurement based on information-theoretic concepts [4]. This approach, in contrast to the mean-square error approach, uses the information gained and lost during measurement to provide intuitive measures of noise and disturbance; that is, it looks at the correlations between input states and measurement outcomes. As for entropic uncertainty relations, this approach is invariant under the relabelling of measurement outcomes and, furthermore, provides measures of noise and disturbance that are state-independent: they depend only on the measurement performed and the complementary observables in question.

In proposing this approach, the authors prove a state-independent measurement uncertainty relation that is valid for arbitrary observables in any finite Hilbert space [4]. However, as is the case with similar preparation uncertainty relations, the result is far from tight in general. It is thus of interest to look at simpler systems to find tight relations and fully understand the noise-noise and noise-disturbance tradeoffs. The simplest nontrivial system one can envisage is, of course, the qubit, and in a subsequent paper an apparently tight noise-disturbance relation for orthogonal qubit observables was proposed and tested experimentally [16]. Unfortunately, as we will discuss, the proof of this relation was incorrect, thus casting doubt on its validity; indeed, we will show that it is incorrect in general, although it can be shown to hold in some particular cases.

In this contribution, we revisit the qubit scenario, looking not only at noise-disturbance relations, but also at noise-noise relations for joint measurements. We completely characterise the joint measurement scenario for arbitrary qubit observables, showing that the set of obtainable noise-noise values is convex and that it seems four-outcomes measurements are required to saturate the tradeoff. On the other hand, we provide evidence that the set of obtainable noise-disturbance points is non-convex, and three-outcome measurements are sufficient to saturate the tradeoff. Finally, we prove that a class of measurements implementing the so-called "square-root dynamics" is not optimal, and thus that non-trivial corrections are needed to perform optimal measurements with respect to the noise-disturbance tradeoff.

2 Theoretical framework

2.1 Entropic definitions of noise and disturbance

Let us first outline the information-theoretic framework for quantifying noise and disturbance that we shall use, and which was first presented in [4]. Recall that we are interested in the degree to which a measurement apparatus $\mathscr{M}$ differs from an "accurate" measurement of an observable A (i.e., the noise) and the extent to which it disturbs a subsequent measurement of a complementary observable B.

Formally, we consider two (for simplicity, non-degenerate) observables A and B of a finite dimensional Hilbert space with respective (normalised) eigenstates $\{|a\rangle\}_a$ and $\{|b\rangle\}_b$, where a and b label the respective eigenvalues (their numerical values are irrelevant).

The measurement device $\mathscr{M}$, with measurement outcomes labelled by m, is represented in the most general way possible as a quantum instrument [7]. Let us briefly recall the definition of a quantum instrument.

Definition 1. *A quantum instrument $\mathscr{M}$ is a collection $\{\mathscr{M}_m\}_m$ of completely positive trace-non-increasing (CP) maps $\mathscr{M}_m$ such that the map*[1] *$\mathscr{M} = \sum_m \mathscr{M}_m$ is a completely positive trace-preserving (CPTP) map, i.e.,* $\mathrm{Tr}[\mathscr{M}(\rho)] = \mathrm{Tr}[\rho]$ *for all Hermitian ρ. The probability of obtaining outcome m when measuring $\mathscr{M}$ on a state ρ is* $\mathrm{Tr}[\mathscr{M}_m(\rho)]$*, and the post-measurement state is $\frac{\mathscr{M}_m(\rho)}{\mathrm{Tr}[\mathscr{M}_m(\rho)]}$ for any state ρ.*

Let us consider the noise of $\mathscr{M}$ with respect to A, $N(\mathscr{M},A)$. Imagine an experiment in which the eigenstates $|a\rangle$ of A are prepared with equal probability and measured by $\mathscr{M}$. The correlation between the eigenvalue a of the state prepared and the outcome m measured, which will be used to define the noise, is characterised by the joint probability distribution

$$p(m,a) = p(a)p(m|a) = \frac{1}{d}\, p(m|a), \tag{1}$$

where d is the Hilbert space dimension, and $p(m|a) = \mathrm{Tr}[\mathscr{M}_m(|a\rangle\langle a|)]$. We denote the classical random variables associated with a and m by $\mathbb{A}$ and $\mathbb{M}$, respectively. This scenario is depicted schematically in Fig. 1(a).

Definition 2. *The noise of $\mathscr{M}$ for a measurement of A is $N(\mathscr{M},A) = H(\mathbb{A}|\mathbb{M})$, where $H(\mathbb{A}|\mathbb{M})$ is the conditional entropy calculated from Eq.* (1).

This definition of noise thus quantifies the uncertainty as to which eigenstate was prepared, given the measurement outcome m of $\mathscr{M}$.

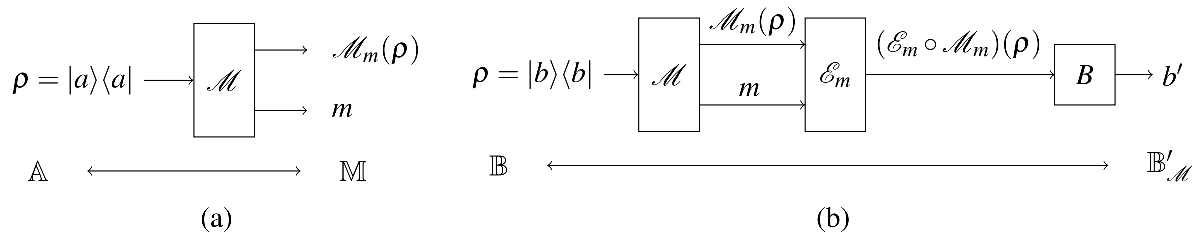


Figure 1: Schematics of the scenarios used in the information-theoretic definitions of (a) noise, $N(\mathscr{M},A)$, and (b) disturbance, $D(\mathscr{M},B)$. The eigenstates $|a\rangle$ of A (or B, for disturbance) are prepared with equal probability, before being measured by $\mathscr{M}$, producing outcome m and transforming the state according to $\mathscr{M}_m$. To calculate the disturbance, as shown in (b), a correction $\mathscr{E}_m$ is then applied and a further projective measurement of B is performed generating the outcome b', which is used to determine the disturbance.

The disturbance $D(\mathscr{M},B)$ is defined with respect to an analogous experiment where this time eigenstates $|b\rangle$ of B are prepared with equal probability, and one looks at the uncertainty in B following the measurement. This is quantified by the correlation between b and the outcome b' of a further projective measurement of B following $\mathscr{M}$. Since the definition should quantify only the irreversible loss of information due to $\mathscr{M}$, a correction $\mathscr{E}_m$ may be performed prior to this subsequent measurement, where $\mathscr{E}_m$ is a CPTP map which may depend on the measurement outcome m. This scenario is characterised by the joint probability distribution

$$p(b',b) = p(b)p(b'|b) = \frac{1}{d}\, p(b'|b), \tag{2}$$

where $p(b'|b)$ is given by the Born rule as

$$p(b'|b) = \mathrm{Tr}\left[(\mathscr{E}\circ\mathscr{M})(|b\rangle\langle b|)\cdot|b'\rangle\langle b'|\right] = \mathrm{Tr}\left[\sum_m (\mathscr{E}_m\circ\mathscr{M}_m)(|b\rangle\langle b|)\cdot|b'\rangle\langle b'|\right]. \tag{3}$$

[1]This slight abuse of notation is generally unambiguous and proves convenient.

We denote the random variables associated with b and b' by $\mathbb{B}$ and $\mathbb{B}'_{\mathscr{M}}$, respectively. This scenario is depicted in Fig. 1(b).

Definition 3. *The disturbance due to* $\mathscr{M}$ *on any subsequent measurement of* B *is* $D(\mathscr{M},B) = \min_{\mathscr{E}} H(\mathbb{B}|\mathbb{B}_{\mathscr{M}})$*, where the minimisation is taken over all correction procedures* $\mathscr{E} = \{\mathscr{E}_m\}_m$ *and the conditional entropy* $H(\mathbb{B}|\mathbb{B}'_{\mathscr{M}})$ *is calculated from Eq.* (2).

2.2 Measurement uncertainty relations

Using these notions of noise and disturbance, Ref. [4] proved that, for arbitrary observables A and B in finite dimensional Hilbert spaces, both the noise-disturbance relation

$$N(\mathscr{M},A) + D(\mathscr{M},B) \geq \log \max_{a,b} |\langle a|b\rangle|^2, \tag{4}$$

and the noise-noise (joint-measurement) relation

$$N(\mathscr{M},A) + N(\mathscr{M},B) \geq \log \max_{a,b} |\langle a|b\rangle|^2, \tag{5}$$

hold. That these relations bear a clear resemblance to the well-known Maassen and Uffink entropic uncertainty relation [13] is no coincidence, since they were proved by showing that

$$N(\mathscr{M},A) + D(\mathscr{M},B) \geq \sum_u p(u)\left[H(A|\rho_u) + H(B|\rho_u)\right] \tag{6}$$

and

$$N(\mathscr{M},A) + N(\mathscr{M},B) = \sum_m p(m)\left[H(A|\rho_m) + H(B|\rho_m)\right] \tag{7}$$

where $\{p(u),\rho_u\}$ and $\{p(m),\rho_m\}$ are specific ensembles of states, and $H(A|\rho)$ is the entropy of A for the state ρ. Thus, the left hand sides of (6) and (7) are bounded below by convex combinations of sums of the entropies of two observables, so that Maassen and Uffink's entropic *preparation* uncertainty relation – which is independent of the state ρ_m – can be applied to each term in the combinations.

However, just like Maassen and Uffink's uncertainty relation, relations (4) and (5) are not tight in general. Rather, one would often like to know precisely which noise-disturbance values are obtainable and which are not; that is, to characterise the *noise-disturbance region*

$$R_{ND}(A,B) = \{(N(\mathscr{M},A),\ D(\mathscr{M},B)) \mid \mathscr{M} \text{ is a quantum instrument}\}, \tag{8}$$

as well as the *noise-noise region*

$$R_{NN}(A,B) = \{(N(\mathscr{M},A),\ N(\mathscr{M},B)) \mid \mathscr{M} \text{ is a quantum instrument}\}. \tag{9}$$

As was shown in the supplementary material of Refs. [4, 16], the lower bound of $R_{ND}(A,B)$ always lies on or above the lower bound of $R_{NN}(A,B)$. More formally, we have the following proposition relating $R_{ND}(A,B)$ and $R_{NN}(A,B)$.

Proposition 4. *For any observables* A,B *one has* $R_{ND}(A,B) \subseteq \mathrm{cl}(R_{NN}(A,B))$*, where* cl *denotes the monotone closure, i.e., the closure under increasing either coordinate.*

Note that it need not be the case that $R_{ND}(A,B) \subseteq R_{NN}(A,B)$ in general. For example, in the scenario depicted in Fig. 2(b), the point $(N(\mathscr{M},A),N(\mathscr{M},B)) = (1,0)$ is not contained in $R_{NN}(A,B)$, whereas $(N(\mathscr{M},A),D(\mathscr{M},B)) = (1,0)$ is always contained in $R_{ND}(A,B)$ since one can have an instrument that performs the identity transformation and generates a random output.

3 Qubit measurement uncertainty relations

The ability to relate these regions to entropic preparation uncertainty relations raises the hope of better understanding and even characterising them for simpler systems, such as qubits. However, obtaining tight noise-disturbance relations is more difficult than simply applying known uncertainty relations to Eq. (6), since the inequality in Eq. (6) is not generally tight and, moreover, one must first characterise the ensemble $\{p(u), \rho_u\}$, which is non-trivial in general.

In Ref. [16] the following noise-disturbance relation was proposed for orthogonal Pauli observables:

$$g\big(N(\mathscr{M}, \sigma_z)\big)^2 + g\big(D(\mathscr{M}, \sigma_x)\big)^2 \leq 1, \tag{10}$$

where g is the inverse of the function h defined for $x \in [0,1]$ as

$$h(x) = -\frac{1+x}{2}\log\left(\frac{1+x}{2}\right) - \frac{1-x}{2}\log\left(\frac{1-x}{2}\right). \tag{11}$$

Unfortunately, the proof given for this relation was incorrect. The approach used, given in the supplementary material of [16], attempts to show first that Eq. (10) characterises the lower bound of $R_{NN}(\sigma_z, \sigma_x)$, before making use of Proposition 4 and the fact that that Eq. (10) can be saturated to show that it is thus also the lower bound of $R_{ND}(\sigma_z, \sigma_x)$.

In the process, they prove the following (correct) result characterising the noise-noise region which we will also make use of later:

Proposition 5. *For Pauli observables $A = \boldsymbol{a}\cdot\boldsymbol{\sigma}$ and $B = \boldsymbol{b}\cdot\boldsymbol{\sigma}$ (where $\boldsymbol{a}, \boldsymbol{b}$ are unit vectors and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$) the noise-noise region $R_{NN}(A,B)$ is given by*

$$\begin{aligned} R_{NN}(A,B) &= \left\{ \sum_m p(m)\,(H(A|\rho_m), H(B|\rho_m)) \mid \{p(m), \rho_m\} \textit{ is } \text{any } \textit{ensemble of states.} \right\} \\ &= \operatorname{conv}\{(H(A|\rho), H(B|\rho)) \mid \rho \textit{ is any density matrix}\}, \end{aligned} \tag{12}$$

where conv S *denotes the convex hull of* S.

Note that for qubits, unlike in the general case expressed in Eq. (7), no restriction needs to be placed on the ensemble in consideration, something not true in general.

Written in this form, it is clear that $R_{NN}(\sigma_z, \sigma_x)$ is a convex set, whereas Eq. (10) characterises a (strictly) concave set (see Fig. 2(a)) and thus cannot be the lower bound of this region, thus undermining the proof given in Ref. [16].

3.1 Joint-measurement uncertainty relations for qubits

Although the approach discussed above does not seem to lead directly to tight noise-disturbance uncertainty relations, this formulation of Proposition 5, when combined with recent results on tight preparation uncertainty relations for qubits, does allows us to formulate tight uncertainty relations for the noise of joint-measurements, i.e., tight noise-noise uncertainty relations, for arbitrary observables on qubits.

Let $A = \boldsymbol{a}.\boldsymbol{\sigma}$ and $B = \boldsymbol{b}.\boldsymbol{\sigma}$ be two arbitrary Pauli observables, and let

$$E(A,B) = \{(H(A|\rho), H(B|\rho)) \mid \rho \text{ is any density matrix}\} \tag{13}$$

so that $R_{NN}(A,B) = \operatorname{conv} E(A,B)$.

In a recent article [1], it was shown that $E(A,B)$ can be completely characterised by the preparation uncertainty relation

$$(\Delta A)^2 + (\Delta B)^2 + 2|\boldsymbol{a}\cdot\boldsymbol{b}|\sqrt{1-(\Delta A)^2}\sqrt{1-(\Delta B)^2} \geq 1 + (\boldsymbol{a}\cdot\boldsymbol{b})^2, \tag{14}$$

or, equivalently, the tight entropic uncertainty relation

$$g\,(H(A))^2 + g\,(H(B))^2 - 2\,|\boldsymbol{a}\cdot\boldsymbol{b}|\,g\,(H(A))\;g\,(H(B)) \leq 1 - (\boldsymbol{a}\cdot\boldsymbol{b})^2. \tag{15}$$

Relation (15), along with Eq. (12) can thus be used to give the following, tight, joint-measurement uncertainty relation for qubits.

Theorem 6. *Let $A = \boldsymbol{a}.\boldsymbol{\sigma}$ and $B = \boldsymbol{b}.\boldsymbol{\sigma}$ be two Pauli observables, and $\mathscr{M}$ an arbitrary quantum measurement. Then the values of $N(\mathscr{M},A)$ and $N(\mathscr{M},B)$ satisfy*

$$R_{NN}(A,B) = \mathrm{conv}\{(u,v) \mid g\,(u)^2 + g\,(v)^2 - 2|\boldsymbol{a}\cdot\boldsymbol{b}|\,g\,(u)\;g\,(v) \leq 1 - (\boldsymbol{a}\cdot\boldsymbol{b})^2\}. \tag{16}$$

Interestingly, the region $E(A,B)$ is non-convex for $|\boldsymbol{a}\cdot\boldsymbol{b}| \lesssim 0.391$ and convex for $|\boldsymbol{a}\cdot\boldsymbol{b}| \gtrsim 0.391$ [15, 17]. Thus, for $|\boldsymbol{a}\cdot\boldsymbol{b}| \gtrsim 0.391$, Eq. (16) can be expressed explicitly as the *tight* uncertainty relation

$$g\,(N(\mathscr{M},A))^2 + g\,(N(\mathscr{M},B))^2 - 2|\boldsymbol{a}\cdot\boldsymbol{b}|\,g\,(N(\mathscr{M},A))\;g\,(N(\mathscr{M},B)) \leq 1 - (\boldsymbol{a}\cdot\boldsymbol{b})^2. \tag{17}$$

For $|\boldsymbol{a}\cdot\boldsymbol{b}| \lesssim 0.391$ a more explicit form may be given, although no analytic form for the convex hull of $E(A,B)$ exists in general. However, for $\boldsymbol{a}\cdot\boldsymbol{b} = 0$, i.e., for orthogonal Pauli measurements, this can be given explicitly and we have the simple tight relation

$$N(\mathscr{M},A) + N(\mathscr{M},B) \geq 1, \tag{18}$$

which is precisely the bound (5) obtained from the Maassen and Uffink uncertainty relation.

The region $R_{NN}(A,B)$ is shown in Figure 2 for two values of $\boldsymbol{a}\cdot\boldsymbol{b}$, along with the region $E(A,B)$.

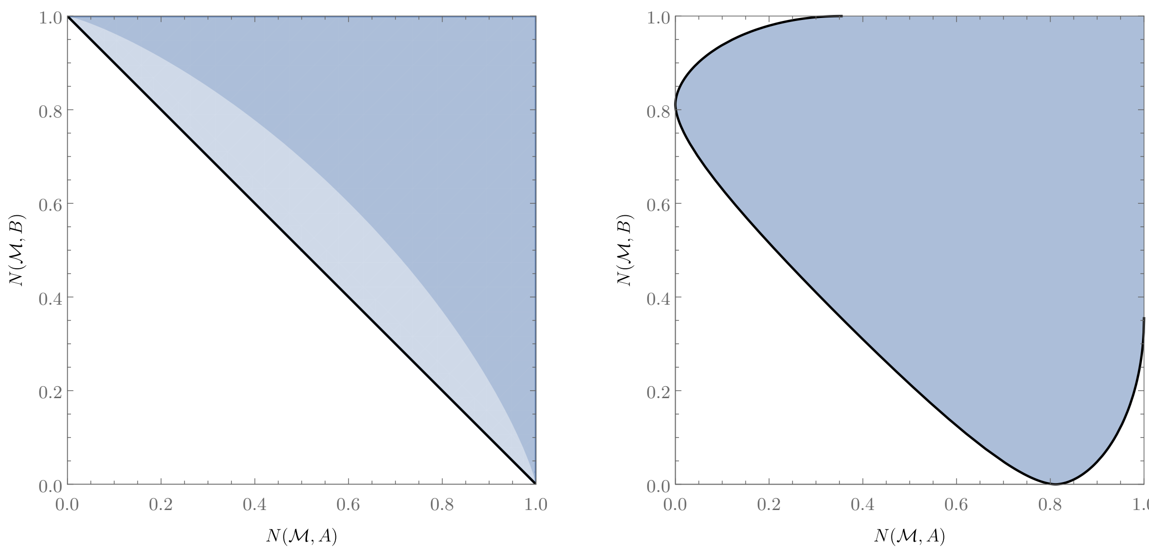
In order to check that the characterisation of $R_{NN}(A,B)$ given in Eq. (16) is tight, one can check that any point $(u,v) \in R_{NN}(A,B) = \mathrm{conv}\,E(A,B)$ can be obtained by some $\mathscr{M}$. Let us first consider the case that $(u,v) \in E(A,B)$. Let $\rho_+ = \frac{1}{2}(\mathbb{1} + \boldsymbol{r}\cdot\boldsymbol{\sigma})$ be a qubit state giving the measurement entropies $(H(A|\rho_+), H(B|\rho_+)) = (u,v)$ and $\rho_- = \frac{1}{2}(\mathbb{1} - \boldsymbol{r}\cdot\boldsymbol{\sigma})$. Then any measurement apparatus $\mathscr{M}$ implementing the positive-operator valued measure (POVM) $\{\rho_+, \rho_-\}$ has $(N(\mathscr{M},A), N(\mathscr{M},B)) = (u,v)$, thus allowing Eq. (17) to be saturated.

To show that any point in $R_{NN}(A,B) \setminus E(A,B)$ can also be obtained (which perhaps corresponds to the case of most interest), we need to make use of POVMs with more outcomes. Since any such point (u,v) is in the convex hull of $E(A,B)$, it can be expressed as a convex combination $q(u_1,v_1) + (1-q)(u_2,v_2)$ of the points $(u_1,v_1), (u_2,v_2) \in E(A,B)$ with $q \in [0,1]$. Let $\{\rho_{1+}, \rho_{1-}\}$ and $\{\rho_{2+}, \rho_{2-}\}$ be two POVMs allowing (u_1,v_1) and (u_2,v_2) to be obtained, respectively, as above. Then an apparatus $\mathscr{M}$ implementing the POVM

$$\{q\rho_{1+},\; q\rho_{1-},\; (1-q)\rho_{2+},\; (1-q)\rho_{2-}\} \tag{19}$$

which performs a combination of these two measurements with probabilities q and $(1-q)$, respectively, gives $(N(\mathscr{M},A), N(\mathscr{M},B)) = (qu_1 + (1-q)u_2, qv_1 + (1-q)v_2) = (u,v)$, thus allowing any point in $R_{NN}(A,B)$ to be realised and, in particular, those on its boundary.

It is interesting to note that, in the case that $E(A,B)$ is not convex, four-outcome POVMs seem not only to be sufficient, but also necessary to obtain the full region $R_{NN}(A,B)$. Numerical simulations generating large numbers of random POVMs appear to show that the region of noise-noise values obtainable with three-outcome POVMs lies in between that obtainable with two-outcome POVMs (which matches $E(A,B)$, see below) and the full region $R_{NN}(A,B)$, but we leave further clarification of this point to future work.



(a) (b)

Figure 2: The shaded area represents the allowable values of $\big(N(\mathscr{M},A),N(\mathscr{M},B)\big)$ for observables $A=\boldsymbol{a}\cdot\boldsymbol{\sigma}$ and $B=\boldsymbol{b}\cdot\boldsymbol{\sigma}$ where (a) $\boldsymbol{a}\cdot\boldsymbol{b}=0$ and (b) $\boldsymbol{a}\cdot\boldsymbol{b}=\frac{1}{2}$. The black line represents the uncertainty relation (16) and the darker shaded area is the entropic preparation uncertainty region $E(A,B)$. Note that, since $E(A,B)$ is convex for $\boldsymbol{a}\cdot\boldsymbol{b}=\frac{1}{2}$, $R_{NN}(A,B)=E(A,B)$, whereas $E(A,B)\subsetneq R_{NN}(A,B)$ for $\boldsymbol{a}\cdot\boldsymbol{b}=0$.

3.2 Noise-disturbance uncertainty relations for qubits

The error in the proof of Eq. (10) given in Ref. [16], along with the differences between the region defined by Eq. (10) and $R_{NN}(\sigma_z,\sigma_x)$, raises the question of whether the noise-disturbance tradeoff can be decreased below Eq. (10).

If we restrict ourselves to measurement apparatuses $\mathscr{M}$ with only two outcomes, it turns out that Eq. (10) does in fact hold true. To show this, we first note that, as shown in the Supplementary Material of Ref. [16], the region $R_{NN}(A,B)$ remains unchanged (in the qubit case) if one requires that the ensemble $\{p(m),\rho_m\}$ appearing in its characterisation (12) must further satisfy[2] $\sum_m p(m)\rho_m=\frac{1}{2}\mathbb{1}$. For dichotomic measurements, if we label the measurement outcomes $\pm$ and write $\rho_\pm=\frac{1}{2}(\mathbb{1}+\boldsymbol{r}_\pm\cdot\boldsymbol{\sigma})$, then this normalisation ensures that $\boldsymbol{r}_+=-\boldsymbol{r}_-$. Since $H(A|\rho_+)=H(A|\rho_-)$, and similarly for B, the restriction of the noise-noise region to two-outcome measurements is simply

$$R^*_{NN}(A,B)=\{(H(A|\rho),H(B|\rho))\mid\rho\text{ is any qubit state}\}=E(A,B). \tag{20}$$

Using a simple generalisation of Proposition 4 one can show that $R^*_{ND}(A,B)\subseteq\mathrm{cl}(R^*_{NN}(A,B))$, where $R^*_{ND}(A,B)$ is the restriction of $R_{ND}(A,B)$ to dichotomic measurements. In the case of orthogonal Pauli observables considered in Ref. [16], when coupled with the facts that Eq. (10) characterises $E(\sigma_z,\sigma_x)$ and can be saturated, this result means that Eq. (10) is indeed tight for dichotomic measurements. This corresponds to the case tested experimentally in Ref. [16], thus showing the inequality they tested was valid (and tight) in their experimental regime.

[2] In fact, such a normalisation constraint is required for all higher-dimensional systems, and can only be ignored for qubits.

The question then remains whether this is the case for measurements with more outcomes, or if it is a peculiarity of dichotomic measurements. In the following section we will show that it does not: there exist measurements which violate Eq. (10). We will then present results towards a tight characterisation of $R_{ND}(\sigma_z,\sigma_x)$, before showing that Eq. (10) nonetheless holds for another important class of measurements.

From here on in, we will restrict ourselves, unless otherwise specified, to the case of orthogonal Pauli observables (i.e., $A=\sigma_z$, $B=\sigma_x$) to which Eq. (10) applies.

3.2.1 Going beyond dichotomic measurements

Consider the three-outcome measurement $\mathscr{M}^\theta$ with the associated POVM $M^\theta=\{M^\theta_{-1},M^\theta_0,M^\theta_1\}$ for $\theta\in[0,\pi/2]$, where $M^\theta_m=\alpha_m(\mathbb{1}+\boldsymbol{n}_m\cdot\boldsymbol{\sigma})$ and $\boldsymbol{n}_m=(\cos(m(\pi+\theta)),0,\sin(m\theta))$, $\alpha_0=\frac{\cos\theta}{1+\cos\theta}$ and $\alpha_{-1}=\alpha_1=\frac{1}{2(1+\cos\theta)}$. One can readily verify that this is a valid POVM, i.e., the M_m are positive semidefinite and $\sum_m M_m=\mathbb{1}$. The probability of obtaining outcome m when measuring a state ρ is thus $\mathrm{Tr}[\rho M_m]$, and we consider the case that, following the measurement, the system is in the pure state $|n_m\rangle$ with Bloch vector $\boldsymbol{n}_m$.

From Eq. (1) we can thus calculate the joint distribution $p(m,a)$ from which we determine the noise on σ_z to be

$$N(\mathscr{M}^\theta,\sigma_z)=\frac{\cos\theta+h(\sin\theta)}{1+\cos\theta}. \tag{21}$$

In order to determine an upper bound on the disturbance $D(\mathscr{M}^\theta,\sigma_x)$, let us consider the correction $\mathscr{E}=\{\mathscr{E}_m\}_m$ that leaves the state unchanged on outcome 0, and maps $\boldsymbol{n}_{-1}$ and $\boldsymbol{n}_1$ onto the negative x-axis. One may implement this with unitary transformations, or, more simply, require that $\mathscr{E}_0(\rho)=\rho$ and $\mathscr{E}_{-1}(\rho)=\mathscr{E}_1(\rho)=\frac{1}{2}(\mathbb{1}-\sigma_x)$ for all ρ. From Eq. (2) one can then calculate the joint distribution $p(b',b)$ and thus the upper-bound on the disturbance as

$$H(\mathbb{X}|\mathbb{X}'_\theta)=\frac{h(\cos\theta)}{1+\cos\theta}, \tag{22}$$

where the subscript on $\mathbb{X}'_\theta$ indicates the random variable $\mathbb{X}'_{\mathscr{M}}$ for the instrument $\mathscr{M}^\theta$.

This measurement-correction pair violates Eq. (10) for all $\theta\in(0,\pi/2)$. Taking, for example, $\theta=\frac{\pi}{3}$ we find $g(N(\mathscr{M}^\theta,\sigma_z))^2+g(H(\mathbb{X}|\mathbb{X}'_\theta))^2\approx 1.1>1$; since $D(\mathscr{M}^\theta,\sigma_x)\le H(\mathbb{X}|\mathbb{X}'_\theta)$ and g is a decreasing function, Eq. (10) is clearly violated.

The bound given parametrically by $(N(\mathscr{M},\sigma_z),D(\mathscr{M},\sigma_x))=\frac{1}{1+\cos\theta}(\cos\theta+h(\sin\theta),h(\cos\theta))$ for $0\le\theta\le\frac{\pi}{2}$ is thus an upper bound for the lower limit of $R_{ND}(\sigma_z,\sigma_x)$. This bound, which is shown in Fig. 3, is asymmetric around the line $N(\mathscr{M},\sigma_z)=D(\mathscr{M},\sigma_x)$, unlike the tight bounds for the joint measurement relations shown in Fig. 2.

3.2.2 Characterising the noise-disturbance region

With this proof that Eq. (10) can be violated, the problem of characterising precisely the lower bound of $R_{NN}(\sigma_z,\sigma_x)$ is opened up once more. Since it seems that the noise-noise and noise-disturbance regions do not coincide, answering this problem requires understanding analytically the tradeoff between noise and disturbance, which is far from trivial.

Perhaps the most immediate problem in attempting such an analysis is the fact that one must minimise over all possible corrections in order to calculate the disturbance for a given measurement. However, by

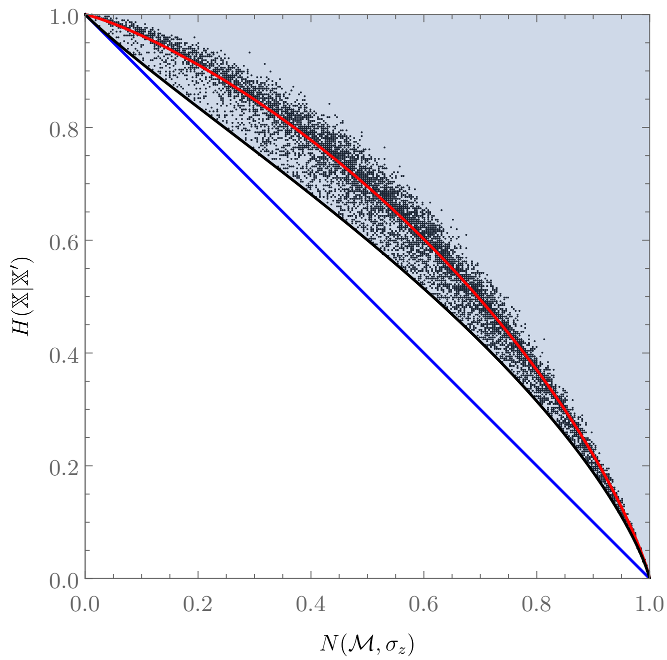


Figure 3: The shaded area represents the conjectured noise-disturbance region (24), the lower bound of which is attained by $\mathscr{M}^\theta$ for $\theta \in [0, \pi/2]$ (black line); the blue line is the noise-noise bound (18); and the red line is the (in general incorrect) bound (10) from Ref. [16]. The noise-disturbance points plotted correspond to ten thousand random three- and four-outcome instruments in the xz-plane with optimal unitary corrections applied.

noting that it is always possible to incorporate a correction into the transformation performed by an instrument to yield another valid instrument, we can see that we restrict ourselves to considering the conditional entropies $H(\mathbb{X}|\mathbb{X}'_{\mathscr{M}})$ in order to characterise the noise-disturbance region. Specifically, we have

$$R_{ND}(\sigma_z, \sigma_x) = \big\{ \big(N(\mathscr{M}, \sigma_z), H(\mathbb{X}|\mathbb{X}'_{\mathscr{M}}) \big) \mid \mathscr{M} \text{ is a quantum instrument} \big\}. \tag{23}$$

Even with this consideration taken into account, the problem remains difficult, and we have not yet been able to prove a tight bound for $R_{ND}(\sigma_z, \sigma_x)$. However, we have performed extensive numerical simulations by testing large numbers of randomly generated quantum instruments with three-or-more outcomes, some of the results of which are shown in Fig. 3. These results suggest that the bound obtained from the counter-example in the previous section is in fact tight. We thus formulate the following conjecture.

Conjecture 7. *Let $\mathscr{M}$ be an arbitrary qubit measurement. Then the values of $N(\mathscr{M}, \sigma_z)$ and $D(\mathscr{M}, \sigma_x)$ satisfy*

$$R_{ND}(\sigma_z, \sigma_x) = \mathrm{cl}\left(\left\{ \tfrac{1}{1+\cos\theta} \big(\cos\theta + h(\sin\theta), h(\cos\theta) \big) \mid 0 \le \theta \le \pi/2 \right\} \right) \cap \{(u,v) \mid 0 \le u, v \le 1\}. \tag{24}$$

This conjecture, if correct, would be surprising since it would indicate that, in stark contrast to the case of joint-measurement noise, three-outcome measurements are sufficient to completely saturate the noise-disturbance bound, and could thus be said to be optimal.

3.2.3 Noise-disturbance relations for square-root dynamics

Although Eq. (10) does not hold in general, our simulations showed that it required carefully chosen post-measurement corrections in order to violate it. In this section we go further and show that it is in fact valid for an interesting class measurements, in which the state is updated according to the so-called "square-root dynamics" [2] and no further correction is applied (i.e., a more restrictive definition of disturbance must be used). A measurement $\mathscr{M}$ with associated POVM $\{M_m\}_m$ is said to implement the square-root dynamics if the state is updated according to $\mathscr{M}_m(\rho) = \sqrt{M_m}\rho\sqrt{M_m}$. This corresponds, for instance, to all projective measurements and many realistic experimental situations.

Let $M = \{M_m\}_m$ be an arbitrary qubit POVM as before. Then we can write each M_m as

$$M_m = \alpha_m(\mathbb{1} + \gamma_m \boldsymbol{n}_m \cdot \boldsymbol{\sigma}), \tag{25}$$

where $|\boldsymbol{n}_m| = 1$, $\alpha_m \geq 0$ and $|\gamma_m| \leq 1$. The normalisation of M, i.e. $\sum_m M_m = \mathbb{1}$, is then expressed by the conditions $\sum_m \alpha_m = 1$ and $\sum_m \alpha_m \gamma_m \boldsymbol{n}_m = \boldsymbol{0}$.

Using this representation we find that the noise $N(\mathscr{M},\sigma_z)$ for any instrument $\mathscr{M}$ realising the POVM M can be expressed as

$$N(\mathscr{M},\sigma_z) = \sum_m \alpha_m h\left(|\gamma_m \boldsymbol{n}_m \cdot \boldsymbol{z}|\right). \tag{26}$$

In order to calculate $H(\mathbb{X}|\mathbb{X}'_{\mathscr{M}})$ we must first calculate the average post-measurement state

$$\rho_+ = \mathscr{M}(|x\rangle\langle x|) = \frac{1}{2}(\mathbb{1} + \boldsymbol{r}_+ \cdot \boldsymbol{\sigma}), \tag{27}$$

as well as the similarly defined $\rho_- = \frac{1}{2}(\mathbb{1} + \boldsymbol{r}_- \cdot \boldsymbol{\sigma})$ for the input $|-x\rangle$. For square-root dynamics, $\mathscr{M}(|x\rangle\langle x|) = \sum_m \sqrt{M_m}|x\rangle\langle x|\sqrt{M_m}$ and we find that

$$\boldsymbol{r}_\pm = \pm\sum_m \alpha_m \left((\boldsymbol{n}_m \cdot \boldsymbol{x})\boldsymbol{n}_m + \sqrt{1-\gamma_m^2}(\boldsymbol{x} - (\boldsymbol{n}_m \cdot \boldsymbol{x})\boldsymbol{n}_m) \right). \tag{28}$$

Letting $\boldsymbol{r} = \boldsymbol{r}_+$ we thus arrive at $H(\mathbb{X}|\mathbb{X}'_{\mathscr{M}}) = h(|\boldsymbol{r}\cdot\boldsymbol{x}|)$.

We will make use of the following fact, which can easily be verified, to show that a measurement following the square-root dynamics (for which the disturbance is simply $H(\mathbb{X}|\mathbb{X}'_{\mathscr{M}})$) must obey Eq. (10).

Fact 8. *The function $f(x) = h(\sqrt{1-x^2})$ is convex on $[0,1]$.*

Theorem 9. *Let $\mathscr{M}$ be a qubit measurement implementing the square-root dynamics. If no additional correction is applied, then $g(N(\mathscr{M},\sigma_z))^2 + g(H(\mathbb{X}|\mathbb{X}'_{\mathscr{M}}))^2 \leq 1$.*

Proof. Let us write the $\boldsymbol{r}$ above as $\boldsymbol{r} = \sum_m \alpha_m \boldsymbol{r}_m$, where

$$\boldsymbol{r}_m = (\boldsymbol{n}_m \cdot \boldsymbol{x})\boldsymbol{n}_m + \sqrt{1-\gamma_m^2}\big(\boldsymbol{x} - (\boldsymbol{n}_m \cdot \boldsymbol{x})\boldsymbol{n}_m\big). \tag{29}$$

Let $u_x = \sum_m \alpha_m |\boldsymbol{r}_m|$ and define the vector $\boldsymbol{u} = u_x \boldsymbol{x} + \sqrt{1-u_x^2}\boldsymbol{z}$. Since $\boldsymbol{n}_m$ and $(\boldsymbol{x} - (\boldsymbol{n}_m \cdot \boldsymbol{x})\boldsymbol{n}_m)$ are orthogonal and $|\boldsymbol{x} - (\boldsymbol{n}_m \cdot \boldsymbol{x})\boldsymbol{n}_m|^2 = 1 - (\boldsymbol{n}_m \cdot \boldsymbol{x})^2$ we have $1 - |\boldsymbol{r}_m|^2 = \gamma_m^2(1 - (\boldsymbol{n}_m \cdot \boldsymbol{x})^2)$. Using Eq. (26) along with the fact that h is decreasing and $|\boldsymbol{n}_m \cdot \boldsymbol{z}| \leq \sqrt{1 - |\boldsymbol{n}_m \cdot \boldsymbol{x}|^2}$, we have

$$\begin{aligned} N(\mathscr{M},\sigma_z) &= \sum_m \alpha_m h\big(|\gamma_m(\boldsymbol{n}_m \cdot \boldsymbol{z})|\big) \geq \sum_m \alpha_m h\left(|\gamma_m|\sqrt{1-(\boldsymbol{n}_m \cdot \boldsymbol{x})^2}\right) = \sum_m \alpha_m h\left(\sqrt{1-|\boldsymbol{r}_m|^2}\right) \\ &\geq h\left(\sqrt{1-(\textstyle\sum_m \alpha_m |\boldsymbol{r}_m|)^2}\right) = h\left(\sqrt{1-u_x^2}\right) = h(\boldsymbol{u}\cdot\boldsymbol{z}) = H(\sigma_z|\boldsymbol{u}), \end{aligned} \tag{30}$$

where we have used Fact 8 to give the second inequality.

Calculating the disturbance for the square-root dynamics, i.e. $H(\mathbb{X}|\mathbb{X}'_{\mathscr{M}})$, we have

$$H(\mathbb{X}|\mathbb{X}'_{\mathscr{M}}) = h\big(|\sum_m \alpha_m \boldsymbol{r}_m \cdot \boldsymbol{x}|\big) \geq h\big(\sum_m \alpha_m |\boldsymbol{r}_m|\big) = h(u_x) = h(\boldsymbol{u}\cdot\boldsymbol{x}) = H(\sigma_x|\boldsymbol{u}). \tag{31}$$

We thus see that the noise and disturbance under the square-root dynamics can both be expressed as the entropy of σ_z and σ_x, respectively, for a common state with Bloch vector $\boldsymbol{u}$, and hence we have $(N(\mathscr{M},\sigma_z), H(\mathbb{X}|\mathbb{X}'_{\mathscr{M}})) \in E(\sigma_z,\sigma_x)$, thereby completing the proof. □

The validity of Eq. (10) for measurements following the square-root dynamics is particularly interesting in that it shows that this interesting class of measurements is not optimal. This contrasts with the results of Ref. [2], who found such dynamics to be optimal with respect to different measures of information gain and disturbance. In order to perform an optimal measurement saturating the noise-disturbance tradeoff bound, one thus needs to consider non-trivial corrections,[3] as in the counter-example of Section 3.2.1, or, equivalently, measurements transforming the system according to more complicated dynamics.

4 Conclusions and future research

In this contribution we have used a recently introduced information-theoretic approach to quantifying both the inherent noise in quantum measurements and the disturbance induced in a state by measurement in order to study, in detail, the noise-disturbance tradeoff in qubit measurements. Using recently published tight entropic uncertainty relations for arbitrary qubit observables, we completely characterised the degree to which two incompatible Pauli observables can be jointly measured. These results could readily be extended to more than two observables to give joint-measurement uncertainty relations for three (or more) Pauli observables using the analogous results for entropic preparation uncertainty relations [1].

We then discussed a recently proposed noise-disturbance uncertainty relation for orthogonal qubit measurements. We showed that the proof given for this relation in Ref. [16] was incorrect and provided a counter-example showing that it can be violated by a three-outcome measurement. We provide a class of measurements that we conjecture saturates the noise-disturbance bound, and provide numerical evidence to back this up. Interestingly, this characterisation of the set of allowable noise-disturbance values only require three-outcome measurements, in contrast to the case of joint measurement, where four-outcomes measurements seem to be necessary.

Finally, we showed that an important class of measurement dynamics – the so-called square-root dynamics – satisfy the more restrictive noise-disturbance relation of Ref. [16] and therefore cannot obtain the optimal noise-disturbance tradeoff. Thus, we show that in order to perform optimal measurements with respect to the qubit noise-disturbance tradeoff, one must utilise non-trivial measurements with more than two outcomes, as well as post-measurement corrections to the state.

It is an open question as to whether the noise-noise and noise-disturbance bounds can be simultaneously saturated by a single measurement, and it would be interesting to further compare these results to those known for more traditional root-mean-square error approaches [14, 5]. Furthermore, our results on the noise-disturbance tradeoff apply only to orthogonal Pauli measurements, and their generalisation to non-orthogonal measurements is left to future work.

[3] One can show that Theorem 9 holds if a single correction is applied irrespective of the measurement outcome.

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