

Operational meanings of orders of observables defined through quantum set theories with different conditionals

Masanao Ozawa*

Graduate School of Information Science, Nagoya University
Nagoya, Japan

ozawa@is.nagoya-u.ac.jp

In quantum logic there is well-known arbitrariness in choosing a binary operation for conditional. Currently, we have at least three candidates, called the Sasaki conditional, the contrapositive Sasaki conditional, and the relevance conditional. A fundamental problem is to show how the form of the conditional follows from an analysis of operational concepts in quantum theory. Here, we attempt such an analysis through quantum set theory (QST). In this paper, we develop quantum set theory based on quantum logics with those three conditionals, each of which defines different quantum logical truth value assignment. We show that those three models satisfy the transfer principle of the same form to determine the quantum logical truth values of theorems of the ZFC set theory. We also show that the reals in the model and the truth values of their equality are the same for those models. Interestingly, however, the order relation between quantum reals significantly depends on the underlying conditionals. We characterize the operational meanings of those order relations in terms of joint probability obtained by the successive projective measurements of arbitrary two observables. Those characterizations clearly show their individual features and will play a fundamental role in future applications to quantum physics.

1 Introduction

Quantum set theory crosses over two different fields of mathematics, namely, foundations of mathematics and foundations of quantum mechanics, and originated from the methods of forcing introduced by Cohen [5, 6] for the independence proof of the continuum hypothesis and quantum logic introduced by Birkhoff and von Neumann [2]. After Cohen's work, the forcing subsequently became a central method in set theory and also incorporated with various notions in mathematics, in particular, the notion of sheaves [8] and notions of sets in nonstandard logics such as Boolean-valued set theory [1], by which Scott and Solovay [22] reformulated the method of forcing, topos [12], and intuitionistic set theory [9]. As a successor of those attempts, quantum set theory, a set theory based on the Birkhoff-von Neumann quantum logic, was introduced by Takeuti [23], who established the one-to-one correspondence between reals in the model (quantum reals) and quantum observables. Quantum set theory was recently developed by the present author [18, 19] to obtain the transfer principle to determine quantum truth values of theorems of the ZFC set theory, and clarify the operational meaning of the equality between quantum reals, which extends the probabilistic interpretation of quantum theory,

In quantum logic there is well-known arbitrariness in choosing a binary operation for conditional. Hardegree [11] defined a material conditional on an orthomodular lattice as a polynomially definable binary operation satisfying three fundamental requirements, and showed that there are exactly three binary operations satisfying those conditions: the Sasaki conditional, the contrapositive Sasaki conditional, and the relevance conditional. Naturally, a fundamental problem is to show how the form of the conditional

*Supported by JSPS KAKENHI, No. 26247016, No. 15K13456.

follows from an analysis of the operational concept in quantum theory testable by experiments. Here, we attempt such an analysis through quantum set theory. In quantum set theory (QST), the quantum logical truth values of two atomic formulas, equality and membership relations, depend crucially on the choice of conditional. In the previous investigations, we have adopted only the Sasaki conditional, proved the transfer principle to determine quantum truth values of theorems of the ZFC set theory, established the one-to-one correspondence between reals in the model, or “quantum reals”, and quantum observables, and clarified the operational meaning of the equality between quantum reals. In this paper, we study QST based on the above three material conditional together. We construct the universal QST model based on the logic of the projection lattice of a von Neumann algebra with each conditional. Then, we show that this new model satisfies the transfer principle of the same form as the old model based on the Sasaki conditional. We also show that the reals in the model and the truth values of their equality are the same for those three models. Up to this point, those models behave indistinguishably. However, we reveal that the order relation between quantum reals depend crucially on the underlying conditionals. We characterize the operational meanings of those order relations, which turn out closely related to the spectral order introduced by Olson [16] playing a significant role in the topos approach to quantum theory [7], in terms of joint probability of the outcomes of the successive projective measurements of two observables. Those characterizations clarify their individual features and will play a fundamental role in future applications to quantum physics.

2 Preliminaries

2.1 Complete orthomodular lattices

A *complete orthomodular lattice* is a complete lattice \mathcal{Q} with an *orthocomplementation*, a unary operation \perp on \mathcal{Q} satisfying (C1) if $P \leq Q$ then $Q^\perp \leq P^\perp$, (C2) $P^{\perp\perp} = P$, (C3) $P \vee P^\perp = 1$ and $P \wedge P^\perp = 0$, where $0 = \bigwedge \mathcal{Q}$ and $1 = \bigvee \mathcal{Q}$, that satisfies the *orthomodular law* (OM) if $P \leq Q$ then $P \vee (P^\perp \wedge Q) = Q$. In this paper, any complete orthomodular lattice is called a *logic*. A non-empty subset of a logic \mathcal{Q} is called a *subalgebra* iff it is closed under \wedge , \vee , and \perp . A subalgebra \mathcal{A} of \mathcal{Q} is said to be *complete* iff it has the supremum and the infimum in \mathcal{Q} of an arbitrary subset of \mathcal{A} . We refer the reader to Kalmbach [13] for a standard text on orthomodular lattices.

We say that P and Q in a logic \mathcal{Q} *commute*, in symbols $P \downarrow Q$, iff $P = (P \wedge Q) \vee (P \wedge Q^\perp)$. A logic \mathcal{Q} is a Boolean algebra if and only if $P \downarrow Q$ for all $P, Q \in \mathcal{Q}$ [13, pp. 24–25]. For any subset $\mathcal{A} \subseteq \mathcal{Q}$, we denote by \mathcal{A}' the *commutant* of \mathcal{A} in \mathcal{Q} [13, p. 23], i.e.,

$$\mathcal{A}' = \{P \in \mathcal{Q} \mid P \downarrow Q \text{ for all } Q \in \mathcal{A}\}.$$

Then, \mathcal{A}' is a complete subalgebra of \mathcal{Q} . A *sublogic* of \mathcal{Q} is a subset \mathcal{A} of \mathcal{Q} satisfying $\mathcal{A} = \mathcal{A}''$. For any subset $\mathcal{A} \subseteq \mathcal{Q}$, the smallest logic including \mathcal{A} is \mathcal{A}'' called the *sublogic generated by \mathcal{A}* . Then, it is easy to see that a subset \mathcal{A} is a Boolean sublogic, or equivalently a distributive sublogic, if and only if $\mathcal{A} = \mathcal{A}'' \subseteq \mathcal{A}'$.

2.2 Logic on Hilbert spaces

Let \mathcal{H} be a Hilbert space. For any subset $S \subseteq \mathcal{H}$, we denote by S^\perp the orthogonal complement of S . Then, $S^{\perp\perp}$ is the closed linear span of S . Let $\mathcal{C}(\mathcal{H})$ be the set of all closed linear subspaces in \mathcal{H} . With the set inclusion ordering, the set $\mathcal{C}(\mathcal{H})$ is a complete lattice. The operation $M \mapsto M^\perp$ is an orthocomplementation on the lattice $\mathcal{C}(\mathcal{H})$, with which $\mathcal{C}(\mathcal{H})$ is a logic.

Denote by $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators on \mathcal{H} and $\mathcal{Q}(\mathcal{H})$ the set of projections on \mathcal{H} . We define the *operator ordering* on $\mathcal{B}(\mathcal{H})$ by $A \leq B$ iff $(\psi, A\psi) \leq (\psi, B\psi)$ for all $\psi \in \mathcal{H}$. For any $A \in \mathcal{B}(\mathcal{H})$, denote by $\mathcal{R}(A) \in \mathcal{C}(\mathcal{H})$ the closure of the range of A , i.e., $\mathcal{R}(A) = (A\mathcal{H})^{\perp\perp}$. For any $M \in \mathcal{C}(\mathcal{H})$, denote by $\mathcal{P}(M) \in \mathcal{Q}(\mathcal{H})$ the projection operator of \mathcal{H} onto M . Then, $\mathcal{R}\mathcal{P}(M) = M$ for all $M \in \mathcal{C}(\mathcal{H})$ and $\mathcal{P}\mathcal{R}(P) = P$ for all $P \in \mathcal{Q}(\mathcal{H})$, and we have $P \leq Q$ if and only if $\mathcal{R}(P) \subseteq \mathcal{R}(Q)$ for all $P, Q \in \mathcal{Q}(\mathcal{H})$, so that $\mathcal{Q}(\mathcal{H})$ with the operator ordering is also a logic isomorphic to $\mathcal{C}(\mathcal{H})$. Any sublogic of $\mathcal{Q}(\mathcal{H})$ will be called a *logic on \mathcal{H}* . The lattice operations are characterized by $P \wedge Q = \text{weak-lim}_{n \rightarrow \infty} (PQ)^n$, $P^\perp = 1 - P$ for all $P, Q \in \mathcal{Q}(\mathcal{H})$.

Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$. We denote by \mathcal{A}' the *commutant* of \mathcal{A} in $\mathcal{B}(\mathcal{H})$. A self-adjoint subalgebra \mathcal{M} of $\mathcal{B}(\mathcal{H})$ is called a *von Neumann algebra* on \mathcal{H} iff $\mathcal{M}'' = \mathcal{M}$. For any self-adjoint subset $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, \mathcal{A}'' is the von Neumann algebra generated by \mathcal{A} . We denote by $\mathcal{P}(\mathcal{M})$ the set of projections in a von Neumann algebra \mathcal{M} . For any $P, Q \in \mathcal{Q}(\mathcal{H})$, we have $P \circ Q$ iff $[P, Q] = 0$, where $[P, Q] = PQ - QP$. For any subset $\mathcal{A} \subseteq \mathcal{Q}(\mathcal{H})$, we denote by \mathcal{A}' the *commutant* of \mathcal{A} in $\mathcal{Q}(\mathcal{H})$. For any subset $\mathcal{A} \subseteq \mathcal{Q}(\mathcal{H})$, the smallest logic including \mathcal{A} is the logic \mathcal{A}'' called the *logic generated by \mathcal{A}* . Then, a subset $\mathcal{Q} \subseteq \mathcal{Q}(\mathcal{H})$ is a logic on \mathcal{H} if and only if $\mathcal{Q} = \mathcal{P}(\mathcal{M})$ for some von Neumann algebra \mathcal{M} on \mathcal{H} [18, Proposition 2.1].

2.3 Commutators

Marsden [14] has introduced the commutator $\perp\!\!\!\perp(P, Q)$ of two elements P and Q of a logic \mathcal{Q} by

$$\perp\!\!\!\perp(P, Q) = (P \wedge Q) \vee (P \wedge Q^\perp) \vee (P^\perp \wedge Q) \vee (P^\perp \wedge Q^\perp).$$

Bruns and Kalmbach [3] have generalized this notion to finite subsets of \mathcal{Q} by

$$\perp\!\!\!\perp(\mathcal{F}) = \bigvee_{\alpha: \mathcal{F} \rightarrow \{\text{id}, \perp\}} \bigwedge_{P \in \mathcal{F}} P^{\alpha(P)}$$

for all $\mathcal{F} \in \mathcal{P}_\omega(\mathcal{Q})$, where $\mathcal{P}_\omega(\mathcal{Q})$ stands for the set of finite subsets of \mathcal{Q} , and $\{\text{id}, \perp\}$ stands for the set consisting of the identity operation id and the orthocomplementation \perp . Generalizing this notion to arbitrary subsets \mathcal{A} of \mathcal{Q} , Takeuti [23] defined $\perp\!\!\!\perp(\mathcal{A})$ by

$$\perp\!\!\!\perp(\mathcal{A}) = \bigvee \{E \in \mathcal{A}' \mid P_1 \wedge E \circ P_2 \wedge E \text{ for all } P_1, P_2 \in \mathcal{A}\},$$

of any $\mathcal{A} \in \mathcal{P}(\mathcal{Q})$, where $\mathcal{P}(\mathcal{Q})$ stands for the power set of \mathcal{Q} . Takeuti's definition has been reformulated in several more convenient forms [21, 4, 19].

We have the following characterizations of commutators in logics on Hilbert spaces [18, Theorems 2.5, 2.6, Proposition 2.2].

Theorem 2.1. *Let \mathcal{Q} be a logic on \mathcal{H} and let $\mathcal{A} \subseteq \mathcal{Q}$. Then, we have the following relations.*

- (i) $\perp\!\!\!\perp(\mathcal{A}) = \mathcal{P}\{\psi \in \mathcal{H} \mid [P_1, P_2]P_3\psi = 0 \text{ for all } P_1, P_2, P_3 \in \mathcal{A}\}$.
- (ii) $\perp\!\!\!\perp(\mathcal{A}) = \mathcal{P}\{\psi \in \mathcal{H} \mid [A, B]\psi = 0 \text{ for all } A, B \in \mathcal{A}''\}$.

3 Conditionals

In classical logic, the conditional operation \rightarrow is defined by negation \perp and disjunction \vee as $P \rightarrow Q = P^\perp \vee Q$. In quantum logic there is a well-known arbitrariness in choosing a binary operation for conditional. Hardegree [11] defined a *material conditional* on an orthomodular lattice \mathcal{Q} as a polynomially definable binary operation \rightarrow on \mathcal{Q} satisfying the following “minimum implicative conditions”:

(LB) If $P \perp Q$, then $P \rightarrow Q = P^\perp \vee Q$ for all $P, Q \in \mathcal{Q}$.

(E) $P \rightarrow Q = 1$ if and only if $P \leq Q$.

(MP) (*modus ponens*) $P \wedge (P \rightarrow Q) \leq Q$.

(MT) (*modus tollens*) $Q^\perp \wedge (P \rightarrow Q) \leq P^\perp$.

Then, he proved that there are exactly three material conditionals:

(S) (Sasaki conditional) $P \rightarrow_S Q := P^\perp \vee (P \wedge Q)$,

(C) (Contrapositive Sasaki conditional) $P \rightarrow_C Q := (P \vee Q)^\perp \vee Q$,

(R) (Relevance conditional) $P \rightarrow_R Q := (P \wedge Q) \vee (P^\perp \wedge Q) \vee (P^\perp \wedge Q^\perp)$.

We shall denote by \rightarrow_j with $j = S, C, R$ any one of the above material conditionals. Once the conditional \rightarrow_j is specified, the logical equivalence \leftrightarrow_j is defined by

$$P \leftrightarrow_j Q := (P \rightarrow_j Q) \wedge (Q \rightarrow_j P).$$

Then, it is easy to see that we have

$$P \leftrightarrow_S Q = P \leftrightarrow_C Q = P \leftrightarrow_R Q = (P \wedge Q) \vee (P^\perp \wedge Q^\perp).$$

Thus, we write \leftrightarrow for \leftrightarrow_j for all $j = S, C, R$.

In the previous investigations [23, 18, 19] on quantum set theory only the Sasaki arrow was adopted as the conditional. In this paper, we develop a quantum set theory based on the above three conditionals together and show that they equally ensure the transfer principle for quantum set theory. We shall also show that the notions of equality defined through those three are the same, but that the order relations defined through them are different.

We have the following characterizations of conditionals in logics on Hilbert spaces.

Theorem 3.1. *Let \mathcal{Q} be a logic on \mathcal{H} and let $\mathcal{A} \subseteq \mathcal{Q}$. Then, we have the following relations.*

- (i) $P \rightarrow_S Q = \mathcal{P}\{\psi \in \mathcal{H} \mid Q^\perp P \psi = 0\}$.
- (ii) $P \rightarrow_C Q = \mathcal{P}\{\psi \in \mathcal{H} \mid P Q^\perp \psi = 0\}$.
- (iii) $P \rightarrow_R Q = \mathcal{P}\{\psi \in \mathcal{H} \mid Q^\perp P \psi = P Q^\perp \psi = 0\}$.

4 Quantum set theory

We denote by V the universe of the Zermelo-Fraenkel set theory with the axiom of choice (ZFC). Let $\mathcal{L}(\in)$ be the language for first-order theory with equality having a binary relation symbol \in , bounded quantifier symbols $\forall x \in y$, $\exists x \in y$, and no constant symbols. For any class U , the language $\mathcal{L}(\in, U)$ is the one obtained by adding a name for each element of U .

Let \mathcal{Q} be a logic on \mathcal{H} . For each ordinal α , let

$$V_\alpha^{(\mathcal{Q})} = \{u \mid u : \text{dom}(u) \rightarrow \mathcal{Q} \text{ and } (\exists \beta < \alpha) \text{dom}(u) \subseteq V_\beta^{(\mathcal{Q})}\}.$$

The \mathcal{Q} -valued universe $V^{(\mathcal{Q})}$ is defined by

$$V^{(\mathcal{Q})} = \bigcup_{\alpha \in \text{On}} V_\alpha^{(\mathcal{Q})},$$

where On is the class of all ordinals. For every $u \in V^{(\mathcal{Q})}$, the rank of u , denoted by $\text{rank}(u)$, is defined as the least α such that $u \in V_{\alpha+1}^{(\mathcal{Q})}$. It is easy to see that if $u \in \text{dom}(v)$ then $\text{rank}(u) < \text{rank}(v)$.

In what follows \rightarrow_j generally denotes one of the Sasaki conditional \rightarrow_S , the contrapositive Sasaki conditional \rightarrow_C , and the relevance conditional \rightarrow_R . For any $u, v \in V^{(\mathcal{L})}$, the \mathcal{Q} -valued truth values of atomic formulas $u = v$ and $u \in v$ are assigned by the following rules recursive in rank.

- (i) $\llbracket u = v \rrbracket_{j, \mathcal{Q}} = \bigwedge_{u' \in \mathcal{D}(u)} (u(u') \rightarrow_j \llbracket u' \in v \rrbracket_{j, \mathcal{Q}}) \wedge \bigwedge_{v' \in \mathcal{D}(v)} (v(v') \rightarrow_j \llbracket v' \in u \rrbracket_{j, \mathcal{Q}})$.
- (ii) $\llbracket u \in v \rrbracket_{j, \mathcal{Q}} = \bigvee_{v' \in \mathcal{D}(v)} (v(v') \wedge \llbracket u = v' \rrbracket_{j, \mathcal{Q}})$.

To each statement ϕ of $\mathcal{L}(\in, V^{(\mathcal{L})})$ we assign the \mathcal{Q} -valued truth value $\llbracket \phi \rrbracket_{j, \mathcal{Q}}$ by the following rules.

- (iii) $\llbracket \neg \phi \rrbracket_{j, \mathcal{Q}} = \llbracket \phi \rrbracket_{j, \mathcal{Q}}^\perp$.
- (iv) $\llbracket \phi_1 \wedge \phi_2 \rrbracket_{j, \mathcal{Q}} = \llbracket \phi_1 \rrbracket_{j, \mathcal{Q}} \wedge \llbracket \phi_2 \rrbracket_{j, \mathcal{Q}}$.
- (v) $\llbracket \phi_1 \vee \phi_2 \rrbracket_{j, \mathcal{Q}} = \llbracket \phi_1 \rrbracket_{j, \mathcal{Q}} \vee \llbracket \phi_2 \rrbracket_{j, \mathcal{Q}}$.
- (vi) $\llbracket \phi_1 \rightarrow \phi_2 \rrbracket_{j, \mathcal{Q}} = \llbracket \phi_1 \rrbracket_{j, \mathcal{Q}} \rightarrow_j \llbracket \phi_2 \rrbracket_{j, \mathcal{Q}}$.
- (vii) $\llbracket \phi_1 \leftrightarrow \phi_2 \rrbracket_{j, \mathcal{Q}} = \llbracket \phi_1 \rrbracket_{j, \mathcal{Q}} \leftrightarrow \llbracket \phi_2 \rrbracket_{j, \mathcal{Q}}$.
- (viii) $\llbracket (\forall x \in u) \phi(x) \rrbracket_{j, \mathcal{Q}} = \bigwedge_{u' \in \text{dom}(u)} (u(u') \rightarrow_j \llbracket \phi(u') \rrbracket_{j, \mathcal{Q}})$.
- (ix) $\llbracket (\exists x \in u) \phi(x) \rrbracket_{j, \mathcal{Q}} = \bigvee_{u' \in \text{dom}(u)} (u(u') \wedge \llbracket \phi(u') \rrbracket_{j, \mathcal{Q}})$.
- (x) $\llbracket (\forall x) \phi(x) \rrbracket_{j, \mathcal{Q}} = \bigwedge_{u \in V^{(\mathcal{L})}} \llbracket \phi(u) \rrbracket_{j, \mathcal{Q}}$.
- (xi) $\llbracket (\exists x) \phi(x) \rrbracket_{j, \mathcal{Q}} = \bigvee_{u \in V^{(\mathcal{L})}} \llbracket \phi(u) \rrbracket_{j, \mathcal{Q}}$.

A formula in $\mathcal{L}(\in)$ is called a Δ_0 -formula if it has no unbounded quantifiers $\forall x$ or $\exists x$. The following theorem holds.

Theorem 4.1 (Δ_0 -Absoluteness Principle). *For any Δ_0 -formula $\phi(x_1, \dots, x_n)$ of $\mathcal{L}(\in)$ and $u_1, \dots, u_n \in V^{(\mathcal{Q})}$, we have*

$$\llbracket \phi(u_1, \dots, u_n) \rrbracket_{j, \mathcal{Q}} = \llbracket \phi(u_1, \dots, u_n) \rrbracket_{j, \mathcal{Q}(\mathcal{H})}.$$

Henceforth, for any Δ_0 -formula $\phi(x_1, \dots, x_n)$ and $u_1, \dots, u_n \in V^{(\mathcal{Q})}$, we abbreviate $\llbracket \phi(u_1, \dots, u_n) \rrbracket_j = \llbracket \phi(u_1, \dots, u_n) \rrbracket_{j, \mathcal{Q}}$, which is the common \mathcal{Q} -valued truth value in all $V^{(\mathcal{L})}$ such that $u_1, \dots, u_n \in V^{(\mathcal{L})}$.

The universe V can be embedded in $V^{(\mathcal{Q})}$ by the following operation $\check{\nu} : v \mapsto \check{\nu}$ defined by the \in -recursion: for each $v \in V$, $\check{\nu} = \{\check{u} \mid u \in v\} \times \{1\}$. Then we have the following.

Theorem 4.2 (Δ_0 -Elementary Equivalence Principle). *For any Δ_0 -formula $\phi(x_1, \dots, x_n)$ of $\mathcal{L}(\in)$ and $u_1, \dots, u_n \in V$, we have $\langle V, \in \rangle \models \phi(u_1, \dots, u_n)$ if and only if $\llbracket \phi(\check{u}_1, \dots, \check{u}_n) \rrbracket_j = 1$.*

5 Transfer Principle in Quantum Set Theory

The results in this section have been obtained for $j = S$ in Ref. [18]. Here, we generalize them to the case $j = C, R$. For $u \in V^{(\mathcal{Q})}$, we define the *support* of u , denoted by $L(u)$, by transfinite recursion on the rank of u by the relation

$$L(u) = \bigcup_{x \in \text{dom}(u)} L(x) \cup \{u(x) \mid x \in \text{dom}(u)\}.$$

For $\mathcal{A} \subseteq V^{(\mathcal{Q})}$ we write $L(\mathcal{A}) = \bigcup_{u \in \mathcal{A}} L(u)$ and for $u_1, \dots, u_n \in V^{(\mathcal{Q})}$ we write $L(u_1, \dots, u_n) = L(\{u_1, \dots, u_n\})$. Let $\mathcal{A} \subseteq V^{(\mathcal{Q})}$. The *commutator* of \mathcal{A} , denoted by $\underline{\vee}(\mathcal{A})$, is defined by

$$\underline{\vee}(\mathcal{A}) = \perp\!\!\!\perp(L(\mathcal{A})).$$

For any $u_1, \dots, u_n \in V^{(\mathcal{Q})}$, we write $\underline{\vee}(u_1, \dots, u_n) = \underline{\vee}(\{u_1, \dots, u_n\})$.

Let $u \in V^{(\mathcal{Q})}$ and $p \in \mathcal{Q}$. The restriction $u|_p$ of u to p is defined by the following transfinite recursion:

$$\begin{aligned} \text{dom}(u|_p) &= \{x|_p \mid x \in \text{dom}(u)\}, \\ u|_p(x|_p) &= u(x) \wedge p \end{aligned}$$

for any $x \in \text{dom}(u)$. By induction, it is easy to see that if $q, p \in \mathcal{Q}$, then $(u|_p)|_q = u|_{p \wedge q}$ for all $u \in V^{(\mathcal{Q})}$.

We have the following theorem.

Theorem 5.1 (Δ_0 -Restriction Principle). *For any Δ_0 -formula $\phi(x_1, \dots, x_n)$ in $L(\in)$ and $u_1, \dots, u_n \in V^{(\mathcal{Q})}$, if $p \in L(u_1, \dots, u_n)^\dagger$, then $[[\phi(u_1, \dots, u_n)]]_j \wedge p = [[\phi(u_1|_p, \dots, u_n|_p)]]_j \wedge p$.*

Then, for bounded theorems of ZFC we obtain the following transfer principle for any material conditionals \rightarrow_j , which is of the same form as Theorem 4.6 of Ref. [18] obtained for the Sasaki conditional \rightarrow_S .

Theorem 5.2 (Δ_0 -ZFC Transfer Principle). *For any Δ_0 -formula $\phi(x_1, \dots, x_n)$ of $\mathcal{L}(\in)$ and $u_1, \dots, u_n \in V^{(\mathcal{Q})}$, if $\phi(x_1, \dots, x_n)$ is provable in ZFC, then we have*

$$\underline{\vee}(u_1, \dots, u_n) \leq [[\phi(u_1, \dots, u_n)]]_j.$$

6 Real numbers in quantum set theory

Let \mathbf{Q} be the set of rational numbers in V . We define the set of rational numbers in the model $V^{(\mathcal{Q})}$ to be $\check{\mathbf{Q}}$. We define a real number in the model by a Dedekind cut of the rational numbers. More precisely, we identify a real number with the upper segment of a Dedekind cut assuming that the lower segment has no end point. Therefore, the formal definition of the predicate $\mathbf{R}(x)$, “ x is a real number,” is expressed by

$$\begin{aligned} \mathbf{R}(x) &:= \forall y \in x (y \in \check{\mathbf{Q}}) \wedge \exists y \in \check{\mathbf{Q}} (y \in x) \wedge \exists y \in \check{\mathbf{Q}} (y \notin x) \\ &\wedge \forall y \in \check{\mathbf{Q}} (y \in x \leftrightarrow \forall z \in \check{\mathbf{Q}} (y < z \rightarrow z \in x)). \end{aligned} \quad (1)$$

We define $\mathbf{R}_j^{(\mathcal{Q})}$ for $j = S, C, R$ to be the interpretation of the set \mathbf{R} of real numbers in $V^{(\mathcal{Q})}$ under the j -conditional as follows.

$$\mathbf{R}_j^{(\mathcal{Q})} = \{u \in V^{(\mathcal{Q})} \mid \mathcal{D}(u) = \mathcal{D}(\check{\mathbf{Q}}) \text{ and } [[\mathbf{R}(u)]]_j = 1\}.$$

The set $\mathbf{R}_{j, \mathcal{Q}}$ of real numbers in $V^{(\mathcal{Q})}$ under the j -conditional is defined by

$$\mathbf{R}_{j, \mathcal{Q}} = \mathbf{R}_j^{(\mathcal{Q})} \times \{1\}. \quad (2)$$

Proposition 6.1. (i) *For any $u \in V^{(\mathcal{Q})}$ with $\text{dom}(u) = \text{dom}(\check{\mathbf{Q}})$, we have*

$$[[\mathbf{R}(u)]]_j = \bigvee_{y \in \mathbf{Q}} u(\check{y}) \wedge \left(\bigwedge_{y \in \mathbf{Q}} u(\check{y}) \right)^\perp \wedge \bigwedge_{y \in \mathbf{Q}} \left(u(\check{y}) \leftrightarrow \bigwedge_{y < z} u(\check{z}) \right).$$

(ii) $\mathbf{R}_{S, \mathcal{Q}} = \mathbf{R}_{C, \mathcal{Q}} = \mathbf{R}_{R, \mathcal{Q}}$.

From the above, in what follows we will write $\mathbf{R}^{(\mathcal{Q})} = \mathbf{R}_j^{(\mathcal{Q})}$ and $\mathbf{R}_{\mathcal{Q}} = \mathbf{R}_{j,\mathcal{Q}}$.

Let \mathcal{M} be a von Neumann algebra on a Hilbert space \mathcal{H} and let $\mathcal{Q} = \mathcal{P}(\mathcal{M})$. A closed operator A (densely defined) on \mathcal{H} is said to be *affiliated* with \mathcal{M} , in symbols $A \eta \mathcal{M}$, iff $U^*AU = A$ for any unitary operator $U \in \mathcal{M}'$. Let A be a self-adjoint operator (densely defined) on \mathcal{H} and let $A = \int_{\mathbf{R}} \lambda dE^A(\lambda)$ be its spectral decomposition, where $\{E^A(\lambda)\}_{\lambda \in \mathbf{R}}$ is the resolution of identity belonging to A [15, p. 119]. It is well-known that $A \eta \mathcal{M}$ if and only if $E^A(\lambda) \in \mathcal{Q}$ for every $\lambda \in \mathbf{R}$. Denote by $\overline{\mathcal{M}}_{SA}$ the set of self-adjoint operators affiliated with \mathcal{M} . Two self-adjoint operators A and B are said to *commute*, in symbols $A \downarrow B$, iff $E^A(\lambda) \downarrow E^B(\lambda')$ for every pair λ, λ' of reals.

For any $u \in \mathbf{R}^{(\mathcal{Q})}$ and $\lambda \in \mathbf{R}$, we define $E^u(\lambda)$ by

$$E^u(\lambda) = \bigwedge_{\lambda < r \in \mathbf{Q}} u(\check{r}). \quad (3)$$

Then, it can be shown that $\{E^u(\lambda)\}_{\lambda \in \mathbf{R}}$ is a resolution of identity in \mathcal{Q} and hence by the spectral theorem there is a self-adjoint operator $\hat{u} \eta \mathcal{M}$ uniquely satisfying $\hat{u} = \int_{\mathbf{R}} \lambda dE^u(\lambda)$. On the other hand, let $A \eta \mathcal{M}$ be a self-adjoint operator. We define $\tilde{A} \in V^{(\mathcal{Q})}$ by

$$\tilde{A} = \{(\check{r}, E^A(r)) \mid r \in \mathbf{Q}\}. \quad (4)$$

Then, $\text{dom}(\tilde{A}) = \text{dom}(\check{\mathbf{Q}})$ and $\tilde{A}(\check{r}) = E^A(r)$ for all $r \in \mathbf{Q}$. It is easy to see that $\tilde{A} \in \mathbf{R}^{(\mathcal{Q})}$ and we have $(\hat{u})^\sim = u$ for all $u \in \mathbf{R}^{(\mathcal{Q})}$ and $(\tilde{A})^\sim = A$ for all $A \in \overline{\mathcal{M}}_{SA}$. Therefore, the correspondence between $\mathbf{R}^{(\mathcal{Q})}$ and $\overline{\mathcal{M}}_{SA}$ is a one-to-one correspondence. We call the above correspondence the *Takeuti correspondence*. Now, we have the following.

Theorem 6.2. *Let \mathcal{Q} be a logic on \mathcal{H} . The relations*

$$(i) \quad E^A(\lambda) = \bigwedge_{\lambda < r \in \mathbf{Q}} u(\check{r}) \text{ for all } \lambda \in \mathbf{Q},$$

$$(ii) \quad u(\check{r}) = E^A(r) \text{ for all } r \in \mathbf{Q},$$

for all $u = \tilde{A} \in \mathbf{R}^{(\mathcal{Q})}$ and $A = \hat{u} \in \overline{\mathcal{M}}_{SA}$ sets up a one-to-one correspondence between $\mathbf{R}^{(\mathcal{Q})}$ and $\overline{\mathcal{M}}_{SA}$.

For any $r \in \mathbf{R}$, we shall write $\tilde{r} = (r1)^\sim$, where $r1$ is the scalar operator on \mathcal{H} . Then, we have $\text{dom}(\tilde{r}) = \text{dom}(\check{\mathbf{Q}})$ and $\tilde{r}(\check{t}) = \llbracket \check{r} \leq \check{t} \rrbracket_j$, so that we have $L(\tilde{r}) = \{0, 1\}$. Denote by $\mathcal{B}(\mathbf{R}^n)$ the σ -field of Borel subsets of \mathbf{R}^n and $B(\mathbf{R}^n)$ the space of bounded Borel functions on \mathbf{R}^n . For any $f \in B(\mathbf{R})$, the bounded self-adjoint operator $f(X) \in \mathcal{M}$ is defined by $f(X) = \int_{\mathbf{R}} f(\lambda) dE^X(\lambda)$. For any Borel subset Δ in \mathbf{R} , we denote by $E^X(\Delta)$ the spectral projection corresponding to $\Delta \in \mathcal{B}(\mathbf{R})$, i.e., $E^X(\Delta) = \chi_{\Delta}(X)$, where χ_{Δ} is the characteristic function of Δ . Then, we have $E^X(\lambda) = E^X((-\infty, \lambda])$.

Proposition 6.3. *Let $r \in \mathbf{R}$, $s, t \in \mathbf{R}$, and $X \eta \mathcal{M}_{SA}$. For $j = R, C, S$, we have the following relations.*

$$(i) \quad \llbracket \check{r} \in \check{s} \rrbracket_j = \llbracket \check{s} \leq \check{r} \rrbracket_j = E^{s1}(t).$$

$$(ii) \quad \llbracket \check{s} \leq \check{t} \rrbracket_j = \llbracket \check{s} \leq \check{t} \rrbracket_j = E^{s1}(t).$$

$$(iii) \quad \llbracket \check{X} \leq \check{t} \rrbracket_j = E^X(t) = E^X((-\infty, t]).$$

$$(iv) \quad \llbracket \check{t} < \check{X} \rrbracket_j = 1 - E^X(t) = E^X((t, \infty)).$$

$$(v) \quad \llbracket \check{s} < \check{X} \leq \check{t} \rrbracket_j = E^X(t) - E^X(s) = E^X((s, t]).$$

$$(vi) \quad \llbracket \check{X} = \check{t} \rrbracket_j = E^X(t) - \bigvee_{r < t, r \in \mathbf{Q}} E^X(r) = E^X(\{t\}).$$

In what follows, we write $r \wedge s = \min\{r, s\}$ and $r \vee s = \max\{r, s\}$ for any $r, s \in \mathbf{R}$.

The \mathcal{Q} -value of equality $\llbracket u = v \rrbracket_j$ for $u, v \in \mathbf{R}^{(\mathcal{Q})}$ is independent of the choice of the conditional and characterized as follows.

Theorem 6.4. For any $u, v \in \mathbf{R}^{(\mathcal{Q})}$ we have

$$\llbracket u = v \rrbracket_j = \mathcal{P}\{\psi \in \mathcal{H} \mid u(\check{x})\psi = v(\check{x})\psi \text{ for all } x \in \mathbf{Q}\}.$$

Theorem 6.5. For any $u, v \in \mathbf{R}^{(\mathcal{Q})}$ and $\psi \in \mathcal{H}$, the following conditions are all equivalent.

- (i) $\psi \in \mathcal{R}\llbracket u = v \rrbracket_j$.
- (ii) $u(\check{x})\psi = v(\check{x})\psi$ for any $x \in \mathbf{Q}$.
- (iii) $u(\check{x})v(\check{y})\psi = v(\check{x} \wedge \check{y})\psi$ for any $x, y \in \mathbf{Q}$.
- (iv) $\langle u(\check{x})\psi, v(\check{y})\psi \rangle = \|v(\check{x} \wedge \check{y})\psi\|^2$ for any $x, y \in \mathbf{Q}$.

The set $\mathbf{R}_{\mathcal{Q}}$ of real numbers in $V^{(\mathcal{Q})}$ is defined by

$$\mathbf{R}_{\mathcal{Q}} = \mathbf{R}^{(\mathcal{Q})} \times \{1\}.$$

Let A be an observable. For any (complex-valued) bounded Borel function f on \mathbf{R} , we define the observable $f(A)$ by

$$f(A) = \int_{\mathbf{R}} f(\lambda) dE^A(\lambda).$$

We shall denote by $B(\mathbf{R})$ the space of bounded Borel functions on \mathbf{R} . For any Borel set Δ in \mathbf{R} , we define $E^A(\Delta)$ by $E^A(\Delta) = \chi_{\Delta}(A)$, where χ_{Δ} is a Borel function on \mathbf{R} defined by $\chi_{\Delta}(x) = 1$ if $x \in \Delta$ and $\chi_{\Delta}(x) = 0$ if $x \notin \Delta$. For any pair of observables A and B , the *joint probability distribution* of A and B in a state ψ is a probability measure $\mu_{\psi}^{A,B}$ on \mathbf{R}^2 satisfying

$$\mu_{\psi}^{A,B}(\Delta \times \Gamma) = \langle \psi, (E^A(\Delta) \wedge E^B(\Gamma))\psi \rangle$$

for any $\Delta, \Gamma \in \mathcal{B}(\mathbf{R})$. Gudder [10] showed that the joint probability distribution $\mu_{\psi}^{A,B}$ exists if and only if the relation $[E^A(\Delta), E^B(\Gamma)]\psi = 0$ holds for every $\Delta, \Gamma \in \mathcal{B}(\mathbf{R})$.

Theorem 6.6. For any observables (self-adjoint operators) A, B on \mathcal{H} and any state (unit vector) $\psi \in \mathcal{H}$, the following conditions are all equivalent.

- (i) $\psi \in \mathcal{R}\llbracket \tilde{A} = \tilde{B} \rrbracket_j$.
- (ii) $E^A(r)\psi = E^B(r)\psi$ for any $r \in \mathbf{Q}$.
- (iii) $f(A)\psi = f(B)\psi$ for all $f \in B(\mathbf{R})$.
- (iv) $\langle E^A(\Delta)\psi, E^B(\Gamma)\psi \rangle = 0$ for any $\Delta, \Gamma \in \mathcal{B}(\mathbf{R})$ with $\Delta \cap \Gamma = \emptyset$.
- (v) There is the joint probability distribution $\mu_{\psi}^{A,B}$ of A and B in ψ satisfying

$$\mu_{\psi}^{A,B}(\{(a, b) \in \mathbf{R}^2 \mid a = b\}) = 1.$$

Condition (iii) above is adopted as the defining condition for A and B to be perfectly correlated in ψ [17] because of the simplicity of the formulation. Condition (v) justifies our nomenclature calling A and B “perfectly correlated.” By condition (i), quantum logic justifies the assertion that “perfectly correlated” observables actually have the same value in the given state. For further properties and applications of the notion of perfect correlation, we refer the reader to Ref. [17].

7 Order relations on quantum reals

Since the real numbers are defined as the upper segment of Dedekind cuts of rational numbers whose lower segment has no end point, the order relation between two quantum reals $u, v \in \mathbf{R}^{(\mathcal{Q})}$ is defined as

$$u \leq v := (\forall r \in v)[r \in u].$$

Let $\mathcal{M} = \mathcal{Q}''$. For any self-adjoint operators $X, Y \in \mathcal{M}$ we write $X \preceq Y$ iff $E^Y(\lambda) \leq E^X(\lambda)$ for all $\lambda \in \mathbf{R}$. The relation is called the *spectral order*. This order is originally introduced by Olson [16] for bounded operators; for recent results for unbounded operators see [20]. With the spectral order the set $\overline{\mathcal{M}}_{SA}$ is a conditionally complete lattice, but it is not a vector lattice; in contrast to the fact that the usual linear order \leq of self-adjoint operators is a lattice if and only if \mathcal{M} is abelian. The following facts about the spectral order are known [16, 20]:

- (i) The spectral order coincides with the usual linear order on projections and mutually commuting operators.
- (ii) For any $X, Y \in \mathcal{M}_{SA}$, we have $X \preceq Y$ if and only if $X^n \leq Y^n$ for all $n \in \mathbf{N}$.

Proposition 7.1. *For any $X, Y \in \mathcal{M}_{SA}$ and $j = S, C, R$, we have $\llbracket \tilde{X} \leq \tilde{Y} \rrbracket_j = 1$ if and only if $X \preceq Y$.*

Proof. We have

$$\llbracket \tilde{X} \leq \tilde{Y} \rrbracket_j = \llbracket (\forall r \in \tilde{Y})[r \in \tilde{X}] \rrbracket_j = \bigwedge_{r \in \text{dom}(\tilde{Y})} \tilde{Y}(r) \rightarrow_j \llbracket r \in \tilde{X} \rrbracket_j = \bigwedge_{r \in \mathbf{Q}} E^Y(r) \rightarrow_j E^X(r)$$

Thus, the assertion follows from the fact that $E^Y(r) \leq E^X(r)$ if and only if $E^Y(r) \rightarrow_j E^X(r) = 1$. \square

To clarify the operational meaning of the truth value $\llbracket \tilde{X} \leq \tilde{Y} \rrbracket_j$, in what follows we shall confine our attention to the case where \mathcal{H} is finite dimensional. We define the joint probability distribution $P_\psi^{X,Y}(x,y)$ representing the joint probability of obtaining the outcomes $Y = y$ and $X = x$ from the successive projective measurements of Y and X , where the X measurement follows immediately after the Y measurement on the same system prepared in the state ψ just before the Y measurement. Then, it is well-known that $P_\psi^{X,Y}(x,y)$ is determined by

$$P_\psi^{X,Y}(x,y) = \|E^X(x)E^Y(y)\psi\|^2.$$

Analogously, we define the joint probability distribution $P_\psi^{Y,X}(y,x)$ obtained by the projective X measurement and the immediately following Y measurement. Then, we have

$$P_\psi^{Y,X}(y,x) = \|E^Y(y)E^X(x)\psi\|^2.$$

Then, we have the following.

Theorem 7.2. *Let X and Y be observables on a finite dimensional Hilbert space \mathcal{H} and ψ be a state in \mathcal{H} . Then, we have the following.*

- (i) $\psi \in \mathcal{R}(\llbracket \tilde{X} \leq \tilde{Y} \rrbracket_S)$ if and only if $P_\psi^{X,Y}(x,y) = 0$ for any $x, y \in \mathbf{R}$ such that $x > y$.
- (ii) $\psi \in \mathcal{R}(\llbracket \tilde{X} \leq \tilde{Y} \rrbracket_C)$ if and only if $P_\psi^{Y,X}(y,x) = 0$ for any $x, y \in \mathbf{R}$ such that $x > y$.
- (iii) $\psi \in \mathcal{R}(\llbracket \tilde{X} \leq \tilde{Y} \rrbracket_R)$ if and only if $P_\psi^{Y,X}(y,x) = P_\psi^{X,Y}(x,y) = 0$ for any $x, y \in \mathbf{R}$ such that $x > y$.

Proof. Let $\psi \in \mathcal{R}(\llbracket \tilde{X} \leq \tilde{Y} \rrbracket_S)$. From Theorem 3.1 we have $E^X(\lambda)^\perp E^Y(\lambda)\psi = 0$ for any $\lambda \in \mathbf{R}$. Then, it is easy to see that $E^X(\lambda)^\perp E^Y(\{\lambda\})\psi = 0$. Thus, if $x > y$ then we have $P_\psi^{X,Y}(x,y) = \|E^X(\{x\})^\perp E^Y(\{y\})\psi\|^2 = 0$. Conversely, suppose that the last equation holds. Then, we have $E^X(\{x\})^\perp E^Y(\{y\})\psi = 0$ for all $x > y$, so that it easily follows that $E^X(\lambda)^\perp E^Y(\lambda)\psi = 0$ for every $\lambda \in \mathbf{R}$. Thus, assertion (i) follows from Theorem 3.1. The rest of the assertions follow routinely. \square

Note that $P_\psi^{X,Y}(x,y) = 0$ for any $x, y \in \mathbf{R}$ such that $x > y$ if and only if $\sum_{x \leq y} P_\psi^{X,Y}(x,y) = 1$ if and only if the outcome of the X -measurement is less than or equal to the outcome of the Y -measurement in a successive (Y, X) -measurement with probability 1. Similarly, $P_\psi^{Y,X}(y,x) = 0$ for any $x, y \in \mathbf{R}$ such that $x > y$ if and only if the outcome of the X -measurement is less than or equal to the outcome of the Y -measurement in a successive (X, Y) -measurement with probability 1.

8 Conclusion

In quantum logic there are at least three candidates for conditional operation, called the Sasaki conditional, the contrapositive Sasaki conditional, and the relevance conditional. In this paper, we have attempted to develop quantum set theory based on quantum logics with those three conditionals, each of which defines different quantum logical truth value assignment. We have shown that those three models satisfy the transfer principle of the same form to determine the quantum logical truth values of theorems of the ZFC set theory. We also show that the reals in the model and the truth values of their equality are the same for those models. Interestingly, however, we have revealed that the order relation between quantum reals significantly depends on the underlying conditionals. In particular, we have completely characterized the operational meanings of those order relations in terms of joint probability obtained by the successive projective measurements of arbitrary two observables. Those characterizations clearly show their individual features and will play a fundamental role in future applications to quantum physics.

References

- [1] J. L. Bell (1985): *Boolean-Valued Models and Independence Proofs in Set Theory*, 2nd edition. Oxford UP, Oxford.
- [2] G. Birkhoff & J. von Neumann (1936): *The Logic of Quantum Mechanics*. *Ann. Math.* 37, pp. 823–843.
- [3] G. Bruns & G. Kalmbach (1973): *Some remarks on free orthomodular lattices*. In J. Schmidt, editor: *Proc. Lattice Theory Conf.*, Houston, U.S.A., pp. 397–408.
- [4] G. Chevalier (1989): *Commutators and Decompositions of Orthomodular Lattices*. *Order* 6, pp. 181–194.
- [5] P. J. Cohen (1963): *The independence of the continuum hypothesis I*. *Proc. Nat. Acad. Sci. U.S.A.* 50, pp. 1143–1148.
- [6] P. J. Cohen (1966): *Set Theory and the Continuum Hypothesis*. Benjamin, New York.
- [7] Andreas Doering & Barry Dewitt (2012): *Self-adjoint Operators as Functions I: Lattices, Galois Connections, and the Spectral Order*. *arXiv preprint arXiv:1208.4724*.
- [8] M. P. Fourman & D. S. Scott (1979): *Sheaves and logic*. In M. P. Fourman, C. J. Mulvey & D. S. Scott, editors: *Applications of Sheaves: Proceedings of the Research Symposium on Applications of Sheaf Theory to Logic, Algebra, and Analysis, Durham, July 9-21, 1977*, Lecture Notes in Math. **753**, Springer, Berlin, pp. 302–401.
- [9] R. J. Grayson (1979): *Heyting-valued models for intuitionistic set theory*. In M. P. Fourman, C. J. Mulvey & D. S. Scott, editors: *Applications of Sheaves: Proceedings of the Research Symposium on Applications of Sheaf Theory to Logic, Algebra, and Analysis, Durham, July 9-21, 1977*, Lecture Notes in Math. 753, Springer, Berlin, pp. 402–414.
- [10] S. Gudder (1968): *Joint Distributions of Observables*. *J. Math. Mech.* 18, pp. 325–335.
- [11] G. M. Hardegree (1981): *Material implication in orthomodular (and Boolean) lattices*. *Notre Dame J. Formal Logic* 22, pp. 163–182.
- [12] P. T. Johnstone (1977): *Topos theory*. London Mathematical Society Monographs, vol. 10, Academic, London.
- [13] G. Kalmbach (1983): *Orthomodular Lattices*. Academic, London.
- [14] E. L. Marsden (1970): *The commutator and solvability in a generalized orthomodular lattice*. *Pacific J. Math* 33, pp. 357–361.
- [15] J. von Neumann (1955): *Mathematical Foundations of Quantum Mechanics*. Princeton UP, Princeton, NJ. [Originally published: *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, 1932)].

- [16] M. P. Olson (1971): *The selfadjoint operators of a von Neumann algebra form a conditionally complete lattice*. *Proc. Amer. Math. Soc.* 28(2), pp. 537–544.
- [17] M. Ozawa (2006): *Quantum perfect correlations*. *Ann. Physics* 321, pp. 744–769.
- [18] M. Ozawa (2007): *Transfer principle in quantum set theory*. *J. Symbolic Logic* 72, pp. 625–648.
- [19] M. Ozawa (2016): *Quantum Set Theory Extending the Standard Probabilistic Interpretation of Quantum Theory*. *New Generat. Comput.* **34**, pp. 125–152. ArXiv:1504.06838 [quant-ph].
- [20] Artur Płaneta & Jan Stochel (2012): *Spectral order for unbounded operators*. *J. Math. Anal. App.* 389(2), pp. 1029–1045.
- [21] S. Pulmannová (1985): *Commutators in orthomodular lattices*. *Demonstratio Math.* 18, pp. 187–208.
- [22] D. Scott & R. Solovay: *Boolean-Valued Models for Set Theory*. Unpublished manuscript for *Proc. AMS Summer Institute on Set Theory*, Los Angeles: Univ. Cal., 1967.
- [23] G. Takeuti (1981): *Quantum set theory*. In E. G. Beltrametti & B. C. van Fraassen, editors: *Current Issues in Quantum Logic*, Plenum, New York, pp. 303–322.