1. In the following graph,

   ![Graph Image]

   a) Find the indicated vertex-deletion subgraphs:

   i) $G - y$

   ![Deletion Subgraph Image]

   ii) $G - \{w,z\}$

   ![Deletion Subgraph Image]
iii) $G - \{w, v, y\}$

b) Find the indicated edge-deletion subgraphs:

i) $G - e$

ii) $G - \{a, m, p, q\}$

iii) $G - \{d, h, k\}$
2. In the following graph,

![Graph Image]

a) Find all the cut-vertices.

Cut-vertices are i and j.

b) Find all the cut-edges (Give edges by its endpoints, i.e. \{a,b\} for the edge with endpoints a and b).

Cut-edges are \{i,j\} and \{j,k\}. (All other edges are cycle-edges.)

3. Find a graph with the vertex-deletion subgraph list given below.

First note that since the list of the vertex-deletion subgraph list is 5, then |V_G|=5. By theorem 2.4.4 (given in presentation on the Graph Reconstruction Problem in class), we know that

\[ |E_G| = \frac{1}{5-2} \sum_{v \in V} |E_{G-v}| = \frac{1}{3} \left( |E_{G-v}| + |E_{G-w}| + |E_{G-x}| + |E_{G-y}| + |E_{G-z}| \right) = \frac{1}{3} (3 + 3 + 3 + 4 + 5) = \frac{1}{3} (18) = 6 \]

And, for each card, the degree of the missing vertex is the difference between \(|E_G|\) and the number of edges on that card. Thus,
\[ \text{deg}(v) = |E_G| - |E_{G,v}| = 6 - 3 = 3. \]
\[ \text{deg}(w) = |E_G| - |E_{G,w}| = 6 - 3 = 3. \]
\[ \text{deg}(x) = |E_G| - |E_{G,x}| = 6 - 3 = 3. \]
\[ \text{deg}(y) = |E_G| - |E_{G,y}| = 6 - 4 = 2. \]
\[ \text{deg}(z) = |E_G| - |E_{G,z}| = 6 - 5 = 1. \]

Thus, the graph has degree sequence \(<3, 3, 3, 2, 1>\) and is isomorphic to

4. **True or false: The endpoints of a cut-edge are both cut-vertices. If false, explain why.**

False. Counterexample: Take any tree of at least two vertices. Every edge is a cut-edge. However, the leaves are not cut-vertices. So any edge with a leaf as an endpoint is a counterexample.

5. **Draw a 6-vertex connected graph that has exactly seven edges and exactly three cycles.**

   or
6. **True or false:** There exists a connected \( n \)-vertex simple graph with \( n + 1 \) edges that contains exactly 2 cycles. If true, give an example. If false, explain why not.

For \( 1 \leq n \leq 3 \), this cannot be true. To even contain \( n + 1 \) edges, the graphs would not be simple.

However, for \( n \geq 4 \), it is TRUE! One realization of this follows:
Consider the path on \( n \) vertices, \( P_n \). We know that it has \( n - 1 \) edges. Now add edges between each leaf and another non-adjacent vertex that is not the other leaf (in the example below, the vertex two away). There are exactly 2 cycles and we’ve added 2 edges to a \( P_n \), so we now have \((n-1)+2=n+1\) edges.

7. For the following, draw the specified tree or explain why no such tree can exist.

   a) A 14-vertex binary tree of height 3.

   ![Diagram of a 14-vertex binary tree of height 3]

   b) A 16-vertex binary tree of height 3.

   No such tree can exist. We know (by Thm. 3.2.3) that for a binary tree \( h + 1 \leq |V| \leq \frac{2^{h+1} - 1}{2 - 1} \) where \( h \) is the height of the binary tree. In our case, we are looking for a tree of height 3, so \( 4 \leq |V| \leq \frac{2^4 - 1}{2 - 1} = \frac{16 - 1}{1} = 15 \).

   So, for a binary tree to be of height 3, it may contain at most 15 vertices. Thus, a 16-vertex binary tree of height 3 cannot exist.
c) A ternary tree of height 3 with exactly four vertices.

There is only one, and it looks like this:

8. What is the relationship between the depth of a vertex v in a rooted tree and the number of ancestors of v? Explain your answer.

They are equal. The depth of a vertex is the distance to the root. An ancestor of a vertex v is a vertex on the path from v to the root. Since there is only one unique path between any two vertices, there is only one path from v to the root so this must be the shortest path. So the number of vertices (not v) on this path is the number of ancestors, and the length of this path is the depth. They must be equal.

9. For the following, draw the root tree specified by the given array of parents.

a) \[
\begin{array}{cccccccccc}
\text{vertex} & a & b & c & d & e & f & g & h & i & j \\
\text{parent} & - & a & b & b & b & b & c & c & c \\
\end{array}
\]
10. Specify the rooted tree drawn below with an array of parents.

Array of parents given by:

<table>
<thead>
<tr>
<th>vertex</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
<th>h</th>
<th>i</th>
<th>j</th>
<th>k</th>
</tr>
</thead>
<tbody>
<tr>
<td>parent</td>
<td>-</td>
<td>a</td>
<td>a</td>
<td>b</td>
<td>b</td>
<td>b</td>
<td>c</td>
<td>c</td>
<td>h</td>
<td>h</td>
<td>j</td>
</tr>
</tbody>
</table>