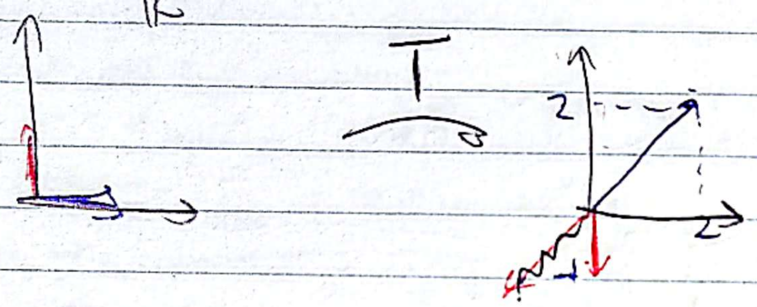
14 Existence of eigenvalues

TPS: 15'
- conditions for eigenvalue & characteristic polynomial;
- polynomials applied to operator & existence of eigenvalues.

TPS Find two l.i. eigenvectors. What are the eigenvalues?



$(x,y) \longmapsto (\cancel{x+2y,2}\ (2x, 2x-y)$

Ⓐ CONDITIONS FOR EIGENVALUE

$Tv=\lambda v \Leftrightarrow (T-\lambda I)v=0 \Leftrightarrow v \in \ker(T-\lambda I)$, $v \neq 0$

Prop TFAE:

(i) λ is an eigenvalue
(ii) $T-\lambda I$ is not injective
(iii) $T-\lambda I$ is not surjective
(iv) $T-\lambda I$ is not invertible $\Leftrightarrow \det(T-\lambda I) \cancel{\neq} = 0$

e.g. TPS: $[T] = \begin{bmatrix} 2 & 0 \\ 2 & -1 \end{bmatrix} \rightarrow [T-\lambda I] = \begin{bmatrix} 2-\lambda & 0 \\ 2 & -1-\lambda \end{bmatrix}$

$0 = \det([T-\lambda I]) = (2-\lambda)(-1-\lambda) \Leftrightarrow \begin{array}{l} \lambda = -1 \\ \lambda = 2 \end{array}$

$$\begin{bmatrix} 2 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -1\begin{bmatrix} x \\ y \end{bmatrix} \leftrightarrow \begin{bmatrix} 2x \\ 2x-y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 3x \\ 2x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{cases} x = 0 \\ y \text{ free variable} \end{cases}$$

e.g. $4\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector w/ eigenvalue -1

$$\begin{bmatrix} 2 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix} \Leftrightarrow \begin{bmatrix} 2x \\ 2x-y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 0 \\ 2x-3y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow y = \frac{3}{2}x$$

e.g. $\begin{bmatrix} 1 \\ 3/2 \end{bmatrix}$ is eigenvector w/ eigenvalue 2

Prop. eigenvectors to distinct eigenvalues are l.i.

Pf We do it for 2 eigenvectors v_1 & v_2 w/eigenvalues λ_1 & λ_2. For the general case induct or see the book.

~~Suppose $v_2 = c\, v_1$ (l.d.). Then~~

Suppose $av_1 + bv_2 = 0$. Then ☆

$$T(av_1 + bv_2) = a\,T(v_1) + b\,T(v_2)$$

$$= \boxed{a\lambda_1 v_1 + b\lambda_2 v_2 = 0}$$

OTOH multiply ☆ by λ_1:

$$\boxed{a\lambda_1 v_1 + b\lambda_1 v_2 = 0}$$

$b\lambda_1 v_2 = b\lambda_2 v_2$

only if $b=0$

$\Downarrow$

$a=0$

l.i.

☒

Cor At most $\dim V$ distinct eigenvalues.

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(B) ~~D~~ EXISTENCE OF EIGENVALUES

Define $T^m = \underbrace{T \circ T \circ \cdots \circ T}_{m \text{ times}}$.

So given a polynomial over $\mathbb{F}$, $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, define $p(T) = a_0 + a_1 T + \cdots + a_n T^n$.

This plays well w/ factorizations:

~~r(z) = s(z)~~

$p(T) \cdot q(T) = (pq)(T)$ [see the book]

e.g. ~~Def the CHARACTERIS~~

Thm T has an eigenvalue

Pf $v, Tv, T^2v, \ldots, T^n v$ is l.d. ($n := \dim V$).

i.e. $\exists$ numbers s.t.

$$a_0 v + a_1 \cdot Tv + a_2 T^2 v + \cdots + a_n T^n v = 0$$

$$\underbrace{(a_0 + a_1 T + a_2 T^2 + \cdots + a_n T^n)}_{p(T)} v = 0$$

(cont. $\to$)

Now consider the same polynomial as an element of $P(\mathbb{C})$

$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$

THIS HAS A ROOT λ. So we can factor

$p(z) = c(z-\lambda) q(z)$

Now

$p(T) = (T-\lambda I) q(T)$

so

$0 = p(T)v = (T-\lambda I) q(T) v \rightarrow T(q(T)v) = \lambda q(T)v$

so $q(T)v$ is an eigenvector w/ eigenvalue λ.

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e.g. rotation by 90° $T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has no real eigenvalues, but yes, two complex eigenvalues (i & $-i$)

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