#15 MINIMAL POLYNOMIAL & UPPER TRIANGULAR MATRICES

- 3b1b video (15') [eigenvectors & eigenvalues]
- minimal polynomial (25')
- upper triangular matrices (40')

(A) MINIMAL POLYNOMIAL

Recall our proof for eigenvalues of $T: V \to V$ over $\mathbb{C}$: we found a polynomial p such that $p(T) = 0$, by finding a root of $p(z)$ and factoring $p(z) = (\lambda - z)\, q(z)$.

It could be the case that $\deg p$ is way bigger than needed since any pol. s.t. $p(T) = 0$ would work for the proof.

Def A monic polynomial is one of the form

$$z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

Thm $\exists!\ m \in \mathcal{P}(\mathbb{C})$ s.t. $m(T) = 0$ w/ smallest degree.

Pf Existence by well-ordering, uniqueness by contradiction.

Def The polynomial above is the <u>minimal polynomial</u> of T

One way to find the minimal polynomial:
Starting at $m=1$ look for sols. of

$$c_0 I + c_1 T + c_2 T^2 + \cdots + c_{m-1} T^{m-1} = -T^m.$$

Could also work evaluating at (almost all) $v \in V$.

e.g. 5.26 in the book..

$\boxed{\text{Prop}}$ Zeroes of $m(T)$ are eigenvalues of T. ①
~~For~~ $F = \mathbb{C} \to m(T) = (z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_m)$ ②

messy

$\boxed{\text{Pf}}$ ① $m(\lambda) = 0 \to m(z) = (z-\lambda)q(z) \to 0 = m(T) = (T-\lambda)q(T)$

$q(T) \neq 0$ (would contradict minimality)
$\to \exists\, v$ st $q(T)v \neq 0$. —— $(T-\lambda)(q(T))(v) = 0$

$T(q(T)v) = \lambda q(T)$ ✓

⊗ If λ is an eigval, e.g. $Tv = \lambda v$. Then $T^k v = \lambda^k v$.
Hence

$0 = m(T)v = m(\lambda)v$
$\quad \hookrightarrow \neq 0$
$\quad \hookrightarrow = 0$..

② Fund. Thm Alg.
minimality removes multiplicity.

Prop $g(T)=0 \to m(T) \mid g(T)$

Pf Book

Thm (Cayley-Hamilton)

Let $c(\lambda) = \det([T-\lambda I])$. Then $c(T)=0$.

In particular ~~m(T)~~ $m(T) \mid c(T)$.

B) UPPER TRIANGULAR MATRICES

A matrix is upper triangular iff it has only zeros below the diagonal. Our goal is to have a better conceptual understanding of these matrices.

→ they are also useful numerically, e.g. $[T][v]=[w]$ can be solved via backwards substitution if $[T]$ is upper triangular e.g. $T: \mathbb{F}^3 \to \mathbb{F}^3$, $T(x,y,z) = (2x+y, 5y+3z, 8z)$

$$[T] = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{bmatrix}$$

Prop TFAE ($T: V \to V$; $e_1, \dots, e_n$ basis)

1. $[T]$ is upper triangular
2. $\mathrm{span}(e_1, \dots, e_k)$ is invariant under T
3. $T v_k \in \mathrm{span}(v_1, \dots, v_k)$

Pf $3 \Rightarrow 1$

$$\begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array} \left[\begin{bmatrix} Te_1 \end{bmatrix} \begin{bmatrix} Te_2 \end{bmatrix} \begin{bmatrix} Te_3 \end{bmatrix} \right]$$

only e_1!

only e_1 & e_2!

only e_1 & e_2 & e_3

etc.