Midterm 1

Matrix Theory & Linear Algebra II

Your name: **Banner ID:**

Assume that you have to explain your reasoning even if the question doesn't explicitly asks you to.

1. (40 points) In this question you are presented with three statements. *Each one of them is wrong.* For each one, give a short (one or two sentences) explanation of why.

   (a) A set of vectors that contains the zero vector can form a basis.

   Solution Let V be a vector space over $\mathbb{F}$. A set containing the zero vector $\mathbf{0} \in V$ is linearly independent, because $c \cdot \mathbf{0} = 0$ for all $c \in \mathbb{F}$. ■

   (b) The unit sphere $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ is a subspace of $\mathbb{R}^3$.

   Solution

   - The sphere doesn't contain the zero vector $(0, 0, 0) \in \mathbb{R}^3$.
   - Vectors in $\mathbb{S}^2$ are not closed under addition. For instance, $(1, 0, 0)$ and $(0, 1, 0)$ are vectors in $\mathbb{S}^2$, but their sum $(1, 0, 0) + (0, 1, 0) = (1, 1, 0)$ is not (it has length $\sqrt{2}$).
   - Vectors in $\mathbb{S}^2$ are not closed under scalar multiplication. Indeed, if you scale the vector $(1, 0, 0)$ in $\mathbb{S}^2$ by a factor of 2 you obtain the vector $(2, 0, 0)$, which is not in $\mathbb{S}^2$ (it has length 2).

   ■

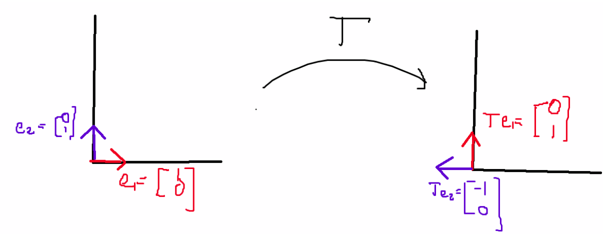
   (c) If v_1, v_2, v_3 spans V, then v_1, v_2, v_3 is a basis for V.

   Solution A basis must also be linearly independent. For instance, the vectors $(1, 0)$, $(1, 1)$ and $(0, 1)$ span $\mathbb{R}^2$, but they are not a basis. ■

2. (30 points) Draw a sketch and find the matrix for each of the following operations on $\mathbb{R}^2$.

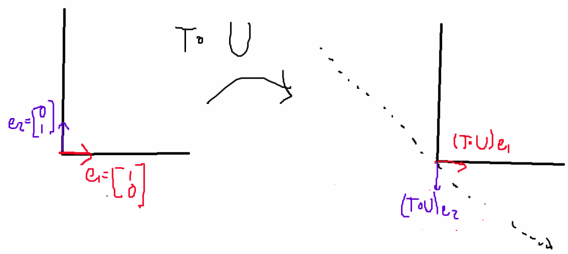
(a) The linear transformation that rotates every vector by an angle of 90°. **errata: counterclockwise**

Solution The transformation is so the corresponding matrix is $[T] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. ■



(b) The linear transformation that reflects every vector about the line $y = -x$.

Solution The transformation is



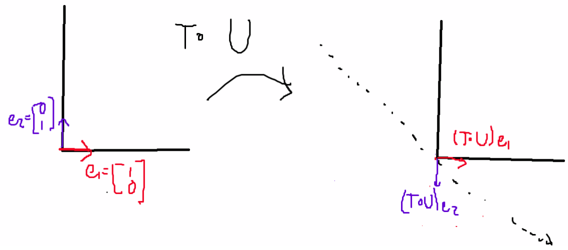
so the corresponding matrix is $[U] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. ■

(c) The linear transformation that performs the operation described in (b), followed by the operation described in (a).

Solution The matrix of $T \circ U$ is the matrix multiplication

$$[T][U] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

From this we can read the transformation visually: *Remark:* this question can also



be done in the reverse order, by first finding the $U \circ T$ visually, then its corresponding matrix. ■

3. (30 points) A linear transformation $P : V \to V$ is a **projection** if $P^2 = P$.

Let $\mathcal{P}_3(\mathbb{R})$ denote the complex vector space of real polynomials with degree at most 3, and consider the linear transformation $U : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ defined by

$$U(a + bx + cx^2 + dx^3) = \frac{a+b}{2} + \frac{a+b}{2}x.$$

(a) Check that U is a projection.

Solution

$$\begin{aligned} U^2(a + bx + cx^2 + dx^3) &= U\left(\underbrace{\frac{a+b}{2}}_{\text{new } a} + \underbrace{\frac{a+b}{2}}_{\text{new } b} \cdot x\right) \\ &= \left(\frac{a+b}{2} + \frac{a+b}{2}\right) + \left(\frac{a+b}{2} + \frac{a+b}{2}\right) \cdot x \\ &= \frac{a+b}{2} + \frac{a+b}{2}x \\ &= U(a + bx + cx^2 + dx^3). \end{aligned}$$

Alternative solution: find the matrix of U in some basis and check that $[U]^2 = [U]$. ■

(b) Find the matrix of U with respect to the basis $1 + x, -1 + x, x^2, x^3$.

Solution we find U applied to each basis vector:

$$\begin{aligned} U(1+x) &= \frac{1+1}{2} + \frac{1+1}{2}x = 1 + x. \\ U(1-x) &= \frac{1-1}{2} + \frac{1-1}{2}x = 0. \\ U(x^2) &= U(x^3) = 0. \end{aligned}$$

The only one of these with non-zero coordinates in this basis is $U(1 + x)$, which has coordinates $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ in this basis. So the matrix of U in this basis is $[U] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

■

(c) What is the dimension of the null space of U? What is the dimension of its range?

Solution The range is spanned by $1 + x$, so the range has dimension 1. Since $4 = \dim \mathcal{P}_3(\mathbb{R}) = \dim U + \dim \operatorname{range} U$, the null space has dimension 3.

Alternative solutions:

- The vectors $1 - x, x^2, x^3$ are a basis for the null space, so the null space has dimension 3. etc.
- The matrix of U has rank 1, so the range has dimension 1. etc.
- The matrix of U has three zero columns, so the null space has dimension 3. etc.

■