Matrix Theory & Linear Algebra II

Practice Midterm I - Solutions

Be careful using these solutions: you can trick yourself into think that you understand more than you do by looking at the explanations Give the practice exam a good try first, and then compare your attempt to these solutions.

1. (a) Complex conjugation doesn't preserve scalar multiplication with complex numbers with imaginary component. For instance

$$\overline{i \cdot (1-i)} = \overline{1+i} = 1-i$$

but

$$i \cdot (\overline{1+i}) = i \cdot (1-i) = 1+i$$

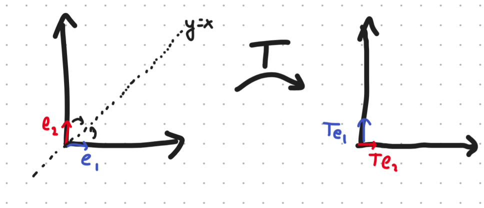
(b) One of the following (or analogous) remarks would suffice.

- The zero matrix $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not in $GL(2, \mathbb{R})$. Indeed, $\det \mathbf{0} = 0$.
- The matrices $\mathbf{A} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ $-\mathbf{A} = \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}$ are in $GL(2, \mathbb{R})$ as $\det \mathbf{A} = \det(-\mathbf{A}) = 25 \neq 0$. However, their sum $\mathbf{A} - \mathbf{A} = \mathbf{0}$ is not.

(c) There are many counterexamples. For instance, let $V = \mathbb{R}^3$, and $v_1 = w_1 = e_1$, $v_2 = w_2 = e_2$, and $v_3 = e_3$, but $w_3 = -e_3$. Then both v_1, v_2, v_3 and w_1, w_2, w_3 are linearly independent lists, but $v_1 + w_1, v_2 + w_2, v_3 + w_3$ is not because $v_3 + w_3$ is the zero vector.

2. In each of these questions, we find the matrix of $S : \mathbb{R}^2 \to \mathbb{R}^2$ as the matrix that has $S(e_1)$ and $S(e_2)$ as its columns, where $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(a) The transformation can be depicted by showing where the basis vectors go:



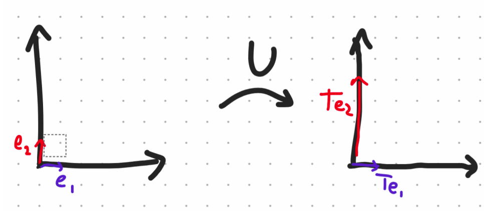
We see that

$$T(e_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad T(e_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

so the matrix of T is

$$[T] = [T(e_1) \mid T(e_2)] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(b) Sketch:



We see that

$$U(e_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad U(e_2) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

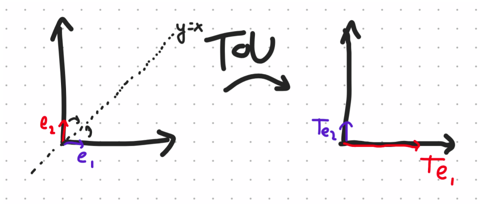
so the matrix of U is

$$[U] = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}.$$

(c) In this item you can draw the sketch, then find the matrix, or proceed as below. The operation is $U \circ T : \mathbb{R}^2 \to \mathbb{R}^2$ so its matrix can be obtained as the matrix composition

$$[T][U] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}.$$

So $T(e_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $T(e_2) = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$. The sketch becomes:



3. (a) We have to check that U each basis element is an eigenvector

- For $i + x$:

$$\begin{aligned} U(i+x) &= (i-1) + (i+1)x && \text{definition of } U \\ &= (i+1)\cdot i + (i+1)x && \text{using the hint} \\ &= (1+i)\cdot(i+x). \end{aligned}$$

- For $i - x$:

$$\begin{aligned} U(-i+x) &= (-i-1) + (-i+1)x && \text{definition of } U \\ &= -(i+1) + (-i+1)x && \text{rearranging} \\ &= i\cdot(i-1) + (-i+1) && \text{using the hint} \\ &= (-i+1)(-i) + (-i+1) && \text{rearranging} \\ &= (1-i)\cdot(-i+x). \end{aligned}$$

- For x^2: $U(x^2) = 0$.

So the matrix of U in the basis $i + x, -i + x, x^2$ is the diagonal matrix

$$[U] = \begin{bmatrix} 1+i & 0 & 0 \\ 0 & 1-i & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(b) *First solution:* note that $U(a + bx + x^2) = 0$ if and only if $a - b = 0$ and $a + b = 0 \implies a = b = 0$. So the null space is spanned by x^2, hence dim null$(U) = 1$. The fundamental theorem of linear transformations says that

$$\dim \mathcal{P}_2(\mathbb{C}) = \dim \operatorname{null}(U) + \dim \operatorname{range}(U).$$

Since dim $\mathcal{P}_2(\mathbb{C}) = 2$, we find out that dim range$(U) = 2$.

Second solution: from the matrix form of U we see two pivots and a column of zeroes. So dim range$(U) = 2$ and dim null$(U) = 1$.