Problem Set 1

Matrix Theory & Linear Algebra II

In this problem set, $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$ (i.e. if $\mathbb{F}$ is in the question, then the solution should be agnostic to whether it was $\mathbb{R}$ or $\mathbb{C}$) and V denotes a vector space over $\mathbb{F}$ (e.g. you can easily check (3) for $\mathbb{F}^n$, but the question requires you to work from the axioms of vector space (or its consequences)).

(1) There are precisely four complex numbers $z \in \mathbb{C}$ such that $z^4 = 1$. What are them?

(2) Does there exist $\lambda \in \mathbb{C}$ such that $\lambda(2 - 3i, 1 + i) = (1 - i, 2)$?

(3) Show that $-(-v) = v$ for any $v \in V$.

(4) Suppose $a \in \mathbb{F}$, $v \in V$, and $av = 0$. Show that $a = 0$ or $v = 0$ (or both).

(5) Consider the set $\mathbb{F}^2$ with the following nonstandard addition operation $\oplus$:

$$(a, b) \oplus (c, d) = (a + d, b + c).$$

Scalar multiplication is defined in the usual way. Is this a vector space? Why?

(6) Is $\mathbb{R}$ naturally a vector space over $\mathbb{C}$? Is $\mathbb{C}$ naturally a vector space over $\mathbb{R}$? *Hint:* only one of the answers is yes.

Remark: the word "natural" means many things in Mathematics. In this case, it means that the operations involved are the ones you would expect. (for instance, the addition operation in the question before this one doesn't feel natural)

(7) Implement $\mathbb{R}^3$ from scratch in your favourite programming language.[1] This means that although your language probably already implements $\mathbb{R}^3$ via arrays like `[a1 a2 a3]`, you should define a type `R3` storing three real numbers, their addition operation, their scalar multiplication, and the zero vector. *Challenge:* do the same for polynomials of dimension at most two. *Hint for the challenge:* it's easy after you do the first part.

The following questions are about subspaces of vector spaces, which we will study on Tuesday Jan 14.

(8) One of the following subsets of $\mathbb{R}^2$ is a subspace and the others are not. For the one that is a vector space, check the conditions for a vector space. For the ones that are not, give a reason: explain one of the axioms for vector space that fails. (There might be more than one!)

(a) The set of pairs $(x, y) \in \mathbb{R}^2$ such that $x = y$.

(b) The set of pairs $(x, y, z) \in \mathbb{R}^2$ such that $x = y^2$.

(c) The set of pairs $(x, y) \in \mathbb{R}^2$ such that y is an integer.

[1] I like using Python via Jupyter. You can install it via Anaconda and run it from the command prompt as `jupyter notebook`.

(d) The set of triples $(x, y) \in \mathbb{R}^2$ such that $x = 1$.

(9) A function is $f : \mathbb{R} \to \mathbb{R}$ is called *periodic* if there exists a positive number p such that $f(x) = f(x + p)$ for all $x \in \mathbb{R}$. Is the set of periodic functions from $\mathbb{R}$ to $\mathbb{R}$ a subspace of $\mathbb{R}^{\mathbb{R}}$? Explain.

(10) Prove the following assertions.

(a) The intersection of two subspaces of V is a subspace.

(b) The union of two subspaces of V is a subspace if and only if one of the subspaces is contained in the other two.

(11) Argue for or against the following statement.

$\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$