# Problem Set 1

## Matrix Theory & Linear Algebra II

In this problem set, $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$ (i.e. if $\mathbb{F}$ is in the question, then the solution should be agnostic to whether it was $\mathbb{R}$ or $\mathbb{C}$) and $V$ denotes a vector space over $\mathbb{F}$ (e.g. you can easily check (3) for $\mathbb{F}^n$, but the question requires you to work from the axioms of vector space (or its consequences)).

(1) There are precisely four complex numbers $z \in \mathbb{C}$ such that $z^4 = 1$. What are them?

*Solution.* The numbers are $1$, $i$, $-1$, and $-i$, and there are many ways you could have found them. Here is one: you could notice that if $z^4 = 1 \implies \sqrt{z^4} = \sqrt{1} = \pm 1$. The roots of $+1$ are $\pm 1$, and the roots of $-1$ are $\pm i$. ∎

(2) Does there exist $\lambda \in \mathbb{C}$ such that $\lambda(2 - 3i, 1 + i) = (1 - i, 2)$?

*Solution.* Let $\lambda = a + bi$. Analyzing the second coordinate, we would need

$$(a + bi)(1 + i) = 2 \iff (a - b) + (a + b)i = 2 \iff \lambda = 1 - i.$$

But in the first coordinate

$$(1 - i)(2 - 3i) \neq 1 - i$$

so there is no solution. ∎

(3) Show that $-(-v) = v$ for any $v \in V$.

*Solution.* Again there are many solutions, but all of them will go through the uniqueness of additive inverses that we proved in class.

One solution: $v - v = 0 \implies v + \underbrace{-v - (-v)}_{=0} = -(-v) \implies v = -(-v)$. Here we just used that $-(-v)$ is (by definition) the additive inverse of $-v$.

Another solution: we saw that $-v = (-1) \cdot v$, so $-(-v) = (-1) \cdot ((-1) \cdot v = ((-1)(-1)) \cdot v = 1 \cdot v = v$ where we used distributivity and unitality of scalar multiplication. ∎

(4) Suppose $a \in \mathbb{F}$, $v \in V$, and $av = 0$. Show that $a = 0$ or $v = 0$ (or both).

*Solution.* If $a \neq 0$ and $a \cdot v = 0$, then we can multiply both sides by $\frac{1}{a}$ to obtain

$$\frac{1}{a} \cdot (a \cdot v) = \frac{1}{a} \cdot 0 \iff \left(\frac{1}{a} \cdot a\right) \cdot v = 0 \iff 1 \cdot v = 0 \iff v = 0.$$

∎

(5) Consider the set $\mathbb{F}^2$ with the following nonstandard addition operation $\oplus$:

$$(a, b) \oplus (c, d) = (a + d, b + c).$$

Scalar multiplication is defined in the usual way. Is this a vector space? Why?

*Solution.* This breaks almost all the axioms. For instance, $(1,0) \oplus (0,1) = (2,0)$ but $(0,1) \oplus (1,0) = (0,2)$ so the operation is not commutative. ∎

(6) Is $\mathbb{R}$ naturally a vector space over $\mathbb{C}$? Is $\mathbb{C}$ naturally a vector space over $\mathbb{R}$? *Hint:* only one of the answers is yes.

*Solution.* You can multiply a complex number $a + bi \in \mathbb{C}$ by a real number $r \in \mathbb{R}$ and still get a complex number:

$$r \cdot (a + bi) = ra + rbi \in \mathbb{C}$$

You can go and check that this turns $\mathbb{C}$ into a vector space over $\mathbb{R}$. (which has $\mathbb{R}$ as a real subspace)

but in general you can't multiply a real number by a complex number and still get a real number, e.g. $i \cdot \pi$ is not a real number. ∎

**Remark:** the word "natural" means many things in Mathematics. In this case, it means that the operations involved are the ones you would expect. (for instance, the addition operation in the question before this one doesn't feel natural)

(7) Implement $\mathbb{R}^3$ from scratch in your favourite programming language.[1] This means that although your language probably already implements $\mathbb{R}^3$ via arrays like `[a1 a2 a3]`, you should define a type `R3` storing three real numbers, their addition operation, their scalar multiplication, and the zero vector. *Challenge:* do the same for polynomials of dimension at most two. *Hint for the challenge:* it's easy after you do the first part.

(8) One of the following subsets of $\mathbb{R}^2$ is a subspace and the others are not. For the one that is a vector space, check the conditions for a vector space. For the ones that are not, give a reason: explain one of the axioms for vector space that fails. (There might be more than one!)

   (a) The set of pairs $(x, y) \in \mathbb{R}^2$ such that $x = y$.
   (b) The set of pairs $(x, y, z) \in \mathbb{R}^2$ such that $x = y^2$.
   (c) The set of pairs $(x, y) \in \mathbb{R}^2$ such that $y$ is an integer.
   (d) The set of triples $(x, y) \in \mathbb{R}^2$ such that $x = 1$.

*Solution.* In (b), (c) and (d) you could find many reasons those sets are not subspaces, I will provide only one.

   (a) . Elements of this set are of the form $(x, x)$ for $x \in \mathbb{R}$, and this is a subspace:
      (i) Given two elements $(x, x)$ and $(y, y)$ in the subset, then their sum $(x, x) + (y, y) = (x + y, x + y)$ is still in the subset.

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[1] I like using Python via Jupyter. You can install it via Anaconda and run it from the command prompt as `jupyter notebook`.
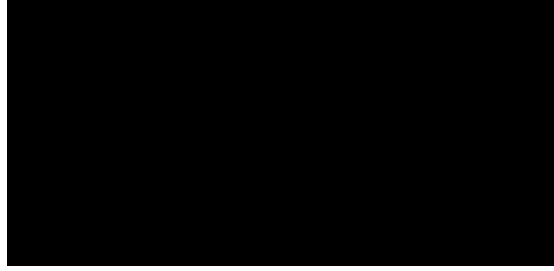
Figure 1: Enter Caption

(ii) Given an element $(x, x)$ and a scalar $a \in \mathbb{R}$, the scaled vector $a \cdot (x, x) = (ax, ax)$ is still in the subset.

(iii) The zero vector $(0, 0)$ is in the subset.

(b) The vectors $(1, 1)$ and $(4, 2)$ are in the subset, but their sum $(1, 1) + (4, 2) = (5, 3)$ is not.

(c) The vector $(1, 1)$ is in the subset, but $\sqrt{2} \cdot (1, 1) = (\sqrt{2}, \sqrt{2})$ is not.

(d) The vectors $(0, 1)$ and $(1, 1)$ are in the subset, but their sum $(0, 1) + (1, 1) = (1, 2)$ is not.

∎

(9) A function is $f : \mathbb{R} \to \mathbb{R}$ is called *periodic* if there exists a positive number $p$ such that $f(x) = f(x + p)$ for all $x \in \mathbb{R}$. Is the set of periodic functions from $\mathbb{R}$ to $\mathbb{R}$ a subspace of $\mathbb{R}^{\mathbb{R}}$? Explain.

*Solution.* The set of periodic functions *with a fixed period* $p$ is a subspace (check the required properties). You can also check that the set of functions with rational periods is a subspace. Read more here: https://math.stackexchange.com/a/3664450/502324.

The only thing that fails for this being a subspace is addition of functions. For instance, the function $f$ that has value 1 at the integers and zero everywhere else is periodic (with period 1). The function $g$ that has value 1 at multiples of $\sqrt{2}$ and zero everywhere else is also periodic (with period 1). Then $(f + g)(0) = 2$, but it is no 2 nowhere else - so it is not periodic!

Indeed, suppose that $(f + g)(a) = 2$ for some non-zero $a$. Then $f(a) + g(a) = 2$. This can only happen if $a$ is an integer which is a multiple of $\sqrt{2}$:

$$a = n \cdot \sqrt{2} \implies \sqrt{2} = \frac{a}{n}.$$

But we know that $\sqrt{2}$ can't be a fraction, so our initial assumption couldn't be right.

∎

(10) Prove the following assertions.

(a) The intersection of two subspaces of $V$ is a subspace.

(b) The union of two subspaces of $V$ is a subspace if and only if one of the subspaces is contained in the other two.

(11) Argue for or against the following statement.

$\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$

*Solution.* There is no correct solution for this one.

You can say that yes, is it a subspace because naturally you can see $(x, y)$ as $(x, y, 0)$.

But I would complain that this is only one way to embed $\mathbb{R}^2 \hookrightarrow \mathbb{R}^3$, there are many other planes in $\mathbb{R}^3$ and there is no way to even ask if it is a subspace without this identification. ∎