Problem Set 2

Matrix Theory & Linear Algebra II

In this problem set, $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$ (i.e. if $\mathbb{F}$ is in the question, then the solution should be agnostic to whether it was $\mathbb{R}$ or $\mathbb{C}$) and V denotes a vector space over $\mathbb{F}$.

(1) Which of the following lists are linearly independent?

(a) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$, as vectors of $\mathbb{R}^3$.

(b) $p_1 = -1, \quad p_2 = x - 1, \quad p_3 = (x-1)^2$, as vectors of $\mathcal{P}(\mathbb{C})$.

(c) $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as vectors of $M_{2,2}(\mathbb{C})$.

(2) A matrix is called *anti-symmetric* if $A^\intercal = -A$. Write down two (distinct!) bases in the space of symmetric 2×2 matrices. How many elements are in each basis?

(3) A polynomial $p \in \mathcal{P}(\mathbb{F})$ is *even* if $p(x) = p(-x)$. Prove that even polynomials form a subspace $U \subseteq \mathcal{P}(\mathbb{F})$ and find a basis for even polynomials in $\mathcal{P}_7(\mathbb{F})$.

(4) Let $U = \{(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^4 \mid 6z_1 = z_2 \text{ and } z_3 + 2z_4 = 0\}$ Find a basis for U, then extend this basis to a basis of $\mathbb{C}^4$.

(5) Let $U = \{p \in \mathcal{P}_4(\mathbb{F}) \mid p(6) = 0\}$. Find a basis for U, then extend this basis to $P_4(\mathbb{F})$.

(6) Prove or give a counterexample: if v_1, v_2, v_3 spans V, then the vectors $w_1 = v_1 - v_2, w_2 = v_2 - v_3$ and $w_3 = v_3 - v_1$ also span V.

(7) Suppose that $v_1, \ldots, v_n$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, \ldots, v_n + w$ is linearly dependent, then $w \in \operatorname{span}(v_1, \ldots, v_n)$.

(8) Prove that $\mathbb{F}^\infty$ is infinite-dimensional. Find a non-zero finite-dimensional subspace.

(9) Prove that the space $C(\mathbb{R})$ of continuous functions $f : \mathbb{R} \to \mathbb{R}$ is infinite-dimensional. *Hint:* you know an infinite-dimensional subspace.

(10) Are the polynomials

$$x^3 - x^2 + 1, x^3 - x^2 + 3, 5x^3 - x^2 + 1, 17x^3 - x^2 + 1 \text{ and } x^2 + 6$$

are linearly independent? *Hint:* dimension.

Now go to chatgpt.com and ask ChatGPT if these polynomials are linearly independent. It will probably get it wrong. When it does, have a conversation with it, and see if you can get it to correct its mistakes.

Note: ChatGPT does not include a logic engine. It tries to answer math questions just by pattern-matching the language, and it tends to agree with whatever you tell it.