Problem Set 2

Matrix Theory & Linear Algebra II

In this problem set, $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$ (i.e. if $\mathbb{F}$ is in the question, then the solution should be agnostic to whether it was $\mathbb{R}$ or $\mathbb{C}$) and V denotes a vector space over $\mathbb{F}$.

(1) Which of the following lists are linearly independent?

(a) $\mathbf{v}_1 = \begin{pmatrix}1\\2\\3\end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix}4\\5\\6\end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix}7\\8\\9\end{pmatrix}$, as vectors of $\mathbb{R}^3$.

(b) $p_1 = -1, \quad p_2 = x - 1, \quad p_3 = (x-1)^2$, as vectors of $\mathcal{P}(\mathbb{C})$.

(c) $\sigma_x = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \quad \sigma_y = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}, \quad \sigma_z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$ as vectors of $M_{2,2}(\mathbb{C})$.

Solution. (a) We have to check whether the equation

$$x\begin{pmatrix}1\\2\\3\end{pmatrix} + y\begin{pmatrix}4\\5\\6\end{pmatrix} + z\begin{pmatrix}7\\8\\9\end{pmatrix} = 0$$

has any non-zero solutions, which correspond to solutions of the system

$$\begin{cases} x + 4y + 7z = 0 \\ 2x + 5y + 8z = 0 \\ 3x + 6y + 9z = 0 \end{cases},$$

which you can find however you'd like. One solution suffices to prove linear dependence, here is an example:

$$1 \cdot \begin{pmatrix}1\\2\\3\end{pmatrix} - 2 \cdot \begin{pmatrix}4\\5\\6\end{pmatrix} + 1 \cdot \begin{pmatrix}7\\8\\9\end{pmatrix} = 0.$$

(b) We have to check whether the equation

$$a \cdot (-1) + b \cdot (x-1) + c \cdot (x-1)^2 = 0$$

has no non-zero solutions. Expanding out and collecting coefficients, the equation becomes

$$(-a - b + c) \cdot 1 + (b - 2c) \cdot x + c \cdot x^2 = 0.$$

A polynomial is zero iff its coefficients are zero (in other words, $\{1, x, x^2\}$ is l.i.). So we have the system

$$\begin{cases} -a - b + c = 0 \\ b - 2c = 0 \\ c = 0 \end{cases}$$

which clearly has only $a = b = c = 0$ as a solution. So the set is linearly independent.

(c) We have to check whether the equation

$$a \cdot \sigma_x + b \cdot \sigma_y + c \cdot \sigma_z = 0$$

has no non-zero solutions. By expanding out coefficients, this corresponds to the equation

$$\begin{pmatrix} c & a - i \cdot b \\ a + i \cdot b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

so $c = 0$ and $a + ib = a - ib = 0 \implies a = b = 0$. So the set is linearly independent.

Remark 1. The matrices σ_x, σ_y and σ_z are called *Pauli matrices* and are fundamental in quantum mechanics.

■

(2) A matrix is called *anti-symmetric* if $A^\intercal = -A$. Write down two (distinct!) bases in the space of **anti-**[1]symmetric 2×2 matrices. How many elements are in each basis?

Solution. A matrix is anti-symmetric if

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -a & -d \\ -b & -d \end{bmatrix}.$$

In particular $a = -a$, so $a = 0$. Similarly $c = 0$ Also $b = -d$.

Summarizing, our vector space is

$$V = \left\{ \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} : b \in \mathbb{F} \right\}.$$

The matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ spans V, and the set $\{A\}$ is trivially linear independent so it forms a basis for V in fact. Another basis is for instance $\left\{ \begin{bmatrix} 0 & 10 \\ -10 & 0 \end{bmatrix} \right\}$. ■

(3) A polynomial $p \in \mathcal{P}(\mathbb{F})$ is *even* if $p(x) = p(-x)$. Prove that even polynomials form a subspace $U \subseteq \mathcal{P}(\mathbb{F})$ and find a basis for even polynomials in $\mathcal{P}_7(\mathbb{F})$.

Solution. Even monic powers $1, x^2, x^4, \ldots, x^{2n}, \ldots$ are even polynomials since

$$(-x)^{2n} = (-1)^{2n} x^{2n} = x^{2n}.$$

Odd powers $x, x^3, x^5, \ldots, x^{2n+1}, \ldots$ are not:

$$(-x)^{2n+1} = (-1)^{2n+1} x^{2n+1} = -x^{2n+1} \neq x^{2n+1}.$$

So a polynomial is even if it is in the span of $1, x^2, x^4, \ldots,$ which is a linearly independent set so it is also a basis for the subspace of even polynomials.

In the case of $\mathcal{P}_7(\mathbb{F})$, a basis for even polynomials is $\{1, x^2, x^4, x^6\}$. ■

[1] This was a typo in the original problem set.

(4) Let[2] $U = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \mid 6z_1 = z_2 \text{ and } z_3 + 2z_4 = 0\}$ Find a basis for U, then extend this basis to a basis of $\mathbb{C}^4$.

Solution. We can write U as the set

$$U = \{(6a, a, -2b, b) \in \mathbb{C}^4 \mid a, b \in \mathbb{C}\}.$$

Since we need two numbers to describe U, it is a good guess that it is two-dimensional. Indeed, the set $\{(6, 1, 0, 0), (0, 0, -2, 1)\}$ is linearly independent (it has just two vectors, which are not multiple of each other) and spans U, so it is a basis for U.

To extend to a basis of U we can add the canonical basis and remove elements, or just add random vectors - you will usually get a set of linearly independent vectors. For instance $\{(6, 1, 0, 0), (0, 0, -2, 1), (1, 0, 0, 0), (0, 0, 1, 0)\}$ is linearly independent (as you should check) and has 4 elements, so it is a basis for $\mathbb{C}^4$. ∎

(5) Let $U = \{p \in \mathcal{P}_4(\mathbb{F}) \mid p(6) = 0\}$. Find a basis for U, then extend this basis to $P_4(\mathbb{F})$.

Solution. We will borrow the observation that

$$\mathcal{B} = \{(x - 6), (x - 6)^2, (x - 6)^3, (x - 6)^4\}$$

is linearly independent set of vectors in U (see question 1b), so the dimension of U is at least 4. Since $P_4(\mathbb{F})$ is 5 dimensional and $1 \notin U$, the dimension of U is at most 4. This implies that $\dim U = 4$ from which we conclude that $\mathcal{B}$ is a basis.

By adding back $1 \in \mathcal{P}(\mathbb{F})$ we get a basis for $\mathcal{P}(\mathbb{F})$:

$$\mathcal{B} = \{1, (x - 6), (x - 6)^2, (x - 6)^3, (x - 6)^4\}$$

∎

(6) Prove or give a counterexample: if v_1, v_2, v_3 spans V, then the vectors $w_1 = v_1 - v_2, w_2 = v_2 - v_3$ and $w_3 = v_3 - v_1$ also span V.

Solution. False, for instance $v_1 = v_2 = v_3 = 1 \in \mathbb{R}$ spans $\mathbb{R}$ (seen as a one-dimensional real vector space), but $v_1 - v_2 = v_2 - v_3 = v_3 - v_1 = 0$ does not. ∎

(7) Suppose that $v_1, \ldots, v_n$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, \ldots, v_n + w$ is linearly dependent, then $w \in \operatorname{span}(v_1, \ldots, v_n)$.

Solution. If $v_1 + w, \ldots, v_n + w$ are linearly dependent then there exists a linear combination

$$c_1 \cdot (v_1 + w) + \cdots + c_n \cdot (v_n + w) = 0$$

where some of the coefficients are non-zero. Rearranging:

$$c_1 \cdot v_1 + \cdots + c_n \cdot v_n = -(c_1 + \cdots + c_n)w$$

The number $-(c_1 + \cdots + c_n)$ has to be non-zero: if it was zero, we would have a non-zero linear combination of $v_1, \ldots, v_n$, contradicting linear independence. So we can write

$$w = -\frac{1}{c_1 + \cdots + c_n}(c_1 \cdot v_1 + \cdots + c_n \cdot v_n),$$

showing that w is in the span of $v_1, \ldots, v_n$. ∎

[2] There was an extra z_5 in the posted problem set.

(8) Prove that $\mathbb{F}^\infty$ is infinite-dimensional. Find a non-zero finite-dimensional subspace.

Solution. Let $a_i \in \mathbb{F}^\infty$ be the sequence with 1 in the i-th entry and 0 everywhere else. For instance

$$a_1 = (1, 0, 0, \dots), \quad a_2 = (0, 1, 0, 0, \dots), \dots$$

These vectors are linearly independent, indeed a linear combination

$$c_1 \cdot a_{k_1} + c_2 \cdot a_{k_2} + \cdots + c_n \cdot a_{k_n} = (0, 0, \dots)$$

implies that $c_1 = c_2 = \cdots = c_n = 0$ by comparing coefficients. So we found a linearly independent set with infinitely many elements, hence any basis has at least infinitely many elements.

Alternate proof: for each natural number n, there is a subspace $\widehat{\mathbb{F}^n} \subseteq \mathbb{F}^\infty$ given by lists with only zeroes after the n-th entry. It's not hard to see that $\widehat{\mathbb{F}^n}$ is n-dimensional. Hence the dimension of $\mathbb{F}^\infty$ is larger than any finite number n, and so itself can't be a finite number. ■

(9) Prove that the space $C(\mathbb{R})$ of continuous functions $f : \mathbb{R} \to \mathbb{R}$ is infinite-dimensional. *Hint:* you know an infinite-dimensional subspace.

Solution. The set of all polynomials forms an infinite-dimensional subspace. ■

(10) Are the polynomials

$$x^3 - x^2 + 1, x^3 - x^2 + 3, 5x^3 - x^2 + 1, 17x^3 - x^2 + 1 \text{ and } x^2 + 6$$

are linearly independent? *Hint:* dimension.

Now go to chatgpt.com and ask ChatGPT if these polynomials are linearly independent. It will probably get it wrong. When it does, have a conversation with it, and see if you can get it to correct its mistakes.

Note: ChatGPT does not include a logic engine. It tries to answer math questions just by pattern-matching the language, and it tends to agree with whatever you tell it.

Solution. The vector space $\mathcal{P}_3(\mathbb{R})$ is 4-dimensional and these are 5 vectors, so the set is linearly dependent. The point of the question was to talk to Chat GPT. (: ■