Problem Set 2

Matrix Theory & Linear Algebra II

Given a vector $v \in V$ and a basis $\mathcal{B} = \{e_1, \dots, e_n\}$ for v, we can find unique coefficient such that

$$v = a_1 e_1 + \cdots + a_n e_n. \tag{1}$$

The numbers $a_1, \dots, a_n$ are the *coordinates* of v and depend on $\mathcal{B}$. We also use the notation

$$[v]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}_{\mathcal{B}}$$

Given a linear transformation $T : V \to W$ and a basis $\mathcal{A} = \{f_1, \dots, f_m\}$ for W, the *matrix of* T is the $m \times n$ matrix

$$[T]_{\mathcal{B},\mathcal{A}} = [[T(e_1)]_{\mathcal{A}} \mid \cdots \mid [T(e_1)]_{\mathcal{A}}].$$

The coordinates of the vector $T(v)$ can be found via matrix-vector multiplication:

$$[T(v)]_{\mathcal{A}} = [T]_{\mathcal{B},\mathcal{A}}[v]_{\mathcal{B}}$$

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(1) Read the handout available at https://people.math.harvard.edu/~knill/teaching/math19b_2011/handouts/lecture08.pdf, then do questions 1 and 2b.

(2) Out of the fours functions from $M_{2,2}(\mathbb{C})$ to $M_{2,2}(\mathbb{C})$, two are linear transformations, and the other two are not. For the ones that are, check the conditions for a linear transformation. For the ones that are not, give a reason: explain one of the axioms for linear transformations that fails. (There might be more than one!)

   (a) The function $f : M_{2,2}(\mathbb{C}) \to M_{2,2}(\mathbb{C})$ defined by $f(A) = A^{\intercal}$.

   (b) The function $f : M_{2,2}(\mathbb{C}) \to M_{2,2}(\mathbb{C})$ defined by $f(A) = MAM^{-1}$, where M is an invertible matrix.

   (c) The function $f : M_{2,2}(\mathbb{C}) \to M_{2,2}(\mathbb{C})$ defined by $f(A) = A^2$.

   (d) The function $f : M_{2,2}(\mathbb{C}) \to M_{2,2}(\mathbb{C})$ defined by $f(A) = A + I$.

(3) Let $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$ be a basis of $\mathbb{R}^2$ and let $x = \begin{bmatrix} 5 \\ -7 \end{bmatrix}$. Find $[x]_{\mathcal{B}}$.

(4) Let $B = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$ be a basis of $\mathbb{R}^3$ and let $x = \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}$ be a vector in $\mathbb{R}^3$. Find $[x]_B$.

(5) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a + b \\ a - b \end{bmatrix}.$$

(a) What is the null space of T? What is its range?

(b) Consider the two bases

$$B_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} \text{ and } B_2 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}.$$

Find the matrix M_{B_2,B_1} of T with respect to the bases B_1 and B_2.

(6) Let $M = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$, and consider the linear transformation $T : M_{2,2}(\mathbb{C}) \to M_{2,2}(\mathbb{C})$ given by $T(A) = MAM$. Find the matrix of T with respect to the basis

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

What is the null space of T? What is its range?

(7) Consider the linear transformation $T : \mathcal{P}_3(\mathbb{F}) \to \mathcal{P}_3(\mathbb{F})$ given by $T(p(x)) = p(x+1)$. Find $[T]_{B,B}$, where $B = \{1, x, x^2, x^3\}$.

(8) Describe the null spaces of the transformations defined in the previous three questions. What is the dimension of their ranges?

(9) Recall that the n-th Taylor polynomial of a differentiable function $f : \mathbb{R} \to \mathbb{R}$, centered around a point $a \in \mathbb{R}$, is the polynomial

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Let $g(x) = 2\sin(x - \pi)$.

(a) Write down an expression for $p_2(x)$ of g centered around π.

(b) What are the coordinates of $p_2(x)$ in the basis $\{1, x, x^2\}$?

(c) What are the coordinates of $p_2(x)$ in the basis $\{1, (x-1), (x-1)^2\}$?

(10) Define an integration linear transformation $\int : \mathcal{P}_n(\mathbb{R}) \to \mathcal{P}_{n+1}$ such that $\frac{d}{dx} \circ \int$ is the identity transformation on $\mathcal{P}_n$.