Problem Set 4

Matrix Theory & Linear Algebra II

(1) Suppose $T \in \mathcal{L}(V, W)$ is invertible. Show that T^{-1} is invertible and

$$(T^{-1})^{-1} = T.$$

(2) Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps. Prove that $ST \in \mathcal{L}(U, W)$ is invertible and that

$$(ST)^{-1} = T^{-1}S^{-1}$$

(3) Show that V and $\mathcal{L}(F, V)$ are isomorphic vector spaces.

(4) Show that $M_{n\times n}(\mathbb{F})$ are isomorphic vector spaces.

(5) Show that $\mathbb{C}$ and $\mathbb{R}^2$ are isomorphic are *real* vector spaces.

(6) True or false:

  (a) Every linear operator in an n-dimensional vector space has n distinct eigenvalues;
  (b) If a matrix has one eigenvector, it has infinitely many eigenvectors;
  (c) There exists a square real matrix with no real eigenvalues;
  (d) There exists a square matrix with no (complex) eigenvectors;
  (e) Similar matrices always have the same eigenvalues;
  (f) Similar matrices always have the same eigenvectors;
  (g) A non-zero sum of two eigenvectors of a matrix A is always an eigenvector;
  (h) A non-zero sum of two eigenvectors of a matrix A corresponding to the same eigenvalue λ is always an eigenvector.

(7) Compute the eigenvalues and eigenvectors of the rotation matrix

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

Note that the eigenvalues (and eigenvectors) do not need to be real.

(8) Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be a basis in a vector space V. Assume also that the first k vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ of the basis are eigenvectors of an operator A, corresponding to an eigenvalue λ (i.e. that $A\mathbf{v}_j = \lambda\mathbf{v}_j, j = 1, 2, \dots, k$). Show that in this basis the matrix of the operator A has block triangular form

$$\begin{pmatrix} \lambda I_k & * \\ 0 & B \end{pmatrix},$$

where I_k is a $k \times k$ identity matrix and B is some $(n-k)\times(n-k)$ matrix.

(9) An operator A is called nilpotent if $A^k = 0$ for some k. Prove that if A is nilpotent, then 0 is the only eigenvalue of A.

(10) Define $T \in \mathcal{L}(\mathbb{C}^3)$ by $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$. Find all eigenvalues and eigenvectors of T.

(11) Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that if λ is an eigenvalue of P, then $\lambda = 0$ or $\lambda = 1$.