Problem Set 5

Matrix Theory & Linear Algebra II

(1) Show that if p is a polynomial with real coefficients and a complex root λ, then $\bar{\lambda}$ is also a root. Conclude that if λ is an eigenvalue of a matrix T with real coefficients, then so is $\bar{\lambda}$.

(2) Find the (real or complex) eigenvalues and eigenvectors of the following matrices. Diagonalize[1] each matrix if possible by finding a basis of eigenvectors.

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

(3) (a) Suppose $\lambda \in \mathbb{F}$ with $\lambda \neq 0$. Prove that λ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

(b) Prove that T and T^{-1} have the same eigenvectors.

(4) Given the matrix:

$$A = \begin{bmatrix} 2 & 6 & -6 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

(a) Find its eigenvalues. Is it possible to find the eigenvalues without computing?

(b) Is this matrix diagonalizable? Find out without computing anything.

(c) If the matrix is diagonalizable, diagonalize it.

(5) Show that if an operator $T : V \to V$ is not invertible, then 0 is an eigenvalue of T.

(6) Let V be a finite-dimensional vector space over $\mathbb{C}$ and let $T \in \mathcal{L}(V)$. Assume that, with respect to some basis (v_1, v_2, v_3, v_4) of V, we have:

$$\mathcal{M}(T) = \begin{pmatrix} 2+i & 3 & -1 & 4 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 2+i & 2 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

(a) Find the eigenvalues of T.

(b) Determine if T is invertible.

(c) Explain why, for each $k \in \{1, 2, 3, 4\}$, we have:

$$T(\text{span}(v_1, \ldots, v_k)) \subseteq \text{span}(v_1, \ldots, v_k).$$

[1] To diagonalize M is to find a diagonal matrix D and a matrix S such that $M = SDS^{-1}$

(7) (a) Give an example of a finite-dimensional complex vector space and an operator T on that vector space such that T^2 is diagonalizable but T is not diagonalizable.

(b) Suppose $\mathbb{F} = \mathbb{C}$, k is a positive integer, and $T \in \mathcal{L}(V)$ is invertible. Prove that T is diagonalizable if and only if T^k is diagonalizable.

(8) Define $T \in \mathcal{L}(P_4(\mathbb{R}))$ by $(Tp)(x) = xp'(x)$ for all $x \in \mathbb{R}$. Find all eigenvalues and eigenvectors of T.

(9) Suppose $T \in \mathcal{L}(V)$ and u, w are eigenvectors of T such that $u + w$ is also an eigenvector of T. Prove that u and w are eigenvectors of T corresponding to the same eigenvalue.

(10) Suppose $T \in \mathcal{L}(V)$ is such that every nonzero vector in V is an eigenvector of T. Prove that T is a scalar multiple of the identity