Problem Set 6

Matrix Theory & Linear Algebra II

Winter 2025

In this problem set, any vector space V comes equipped with an inner product.

(1) Choose one of the inner products defined in Example 6.3 of the book. Check that they are indeed inner products, i.e. show that the axioms are satisfied.

(2) Let $e_1, \dots, e_n$ be an orthonormal basis, and $v = a_1 e_1 + \cdots + a_n v_n$. Show that $a_i = \langle v, e_i \rangle$.

(3) Suppose that $u, v \in V$ and $\|u\| = \|v\| = 1$ and $\langle u, v \rangle = 1$. Prove that $u = v$.

(4) Recall that in $\mathbb{R}^3$ it's true that

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \alpha,$$

where α is the angle between the vectors. There is no "angle" between vectors in a general vector space V, but in the presence of an inner product we can use this formula to define the angle between two vectors:

$$\cos(u, v) = \arccos\left(\frac{\langle u, v \rangle}{\|u\| \|v\|} \right).$$

(a) Show that this formula is well-defined. (What is the domain of arccos?)

(b) Using the inner product $\langle f, g \rangle = \int_0^1 f(x) g(x) dx$ on $C^0([0, 1])$, calculate the angle between x, e^x, and $\sin x$.

(5) Let $\mathcal{B} = e_1, \dots, e_n$ be an orthonormal basis for V, and define the matrix A whose entries are $a_{ij} = \langle e_i, e_j \rangle$. Show that, for $u, v \in V$,

$$\langle u, v \rangle = [u]_{\mathcal{B}}^{\intercal} \cdot A \cdot \overline{[v]}_{\mathcal{B}},$$

where $\overline{[v]}_{\mathcal{B}}$ denotes the vector with the conjugate of coordinates of v as its entries.

(6) In $\mathbb{R}^3$ with the usual dot product, find an orthonormal basis for

$$U = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix} \right).$$

(7) Consider the inner product space $C([0, 2])$ with the inner product given by

$$\langle f, g \rangle = \int_0^2 f(x) g(x) dx.$$

Use Gram-Schmidt to find an orthogonal basis for $\{1, x, x^2\}$.

(8) Consider the vector space $V = C([0, 2\pi], \mathbb{C})$ of continuous functions $f : [0, 2\pi] \to \mathbb{C}$. An element of V is a function of the form

$$f(x) = u(x) + iv(x),$$

where u and v are real valued functions $[0, 2\pi] \to \mathbb{R}$. Suppose $f = u + iv$ and $g = s + it$. Define an inner product of V by

$$\langle f, g \rangle = \int_0^{2\pi} u(x)s(x)dx + i \int_0^{2\pi} v(x)t(x)dx.$$

Calculate the inner product between the functions $e^{ix} = \sin x + i \cos x$, 1, and x.

(9) In $\mathbb{C}^3$ with the complex dot product, find an orthonormal basis for

$$U = \operatorname{span}\left(\begin{bmatrix} 0 \\ i \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ i+1 \\ 3i+2 \end{bmatrix} \right).$$