# MONOIDAL HOMOTOPY BICATEGORIES

in progress (?)

via 2-fibrations

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chosen composites

has higher morphisms, but invertible for n>2

our favourite model are bicategories

(algebraic)

$$\mathrm{Cat}_{(\infty,2)} \xrightarrow{h_2} \mathrm{Cat}_{(2\ \ )}$$

quotients by invertible 3-cells

mostly non-algebraic models

composites live in a contractible space

"only identities" for n>2

[a master's thesis]

usually modelled as algebras over the $E_1$-operad (e.g. [HA])

monoidal bicategories are OK (e.g. [Johnson-Yau, 24'])

$$
\begin{tikzcd}
\mathrm{Cat}_{(\infty,2)}^{\otimes} \arrow[r, "h_2^{\otimes}"] \arrow[r, bend right, "\text{can we describe this directly?}"'] & \mathrm{Cat}_{(2,2)}^{\otimes}
\end{tikzcd}
$$

Why?

- characterize fully dualizable $(\infty,2)$-categories
- give a more direct construction of $h_2^{\otimes}$
- develop monoidal bicategories from $(\infty,2)$-tech

Where do we ?

[Gepner, Haugseng 20'] $\to$ $h_2 : \mathrm{Cat}_{(\infty,2)} \longrightarrow \mathrm{Cat}_{(2,2)}$ as certain truncations

[Romö, 24'] $\to$ $h_2 : CSS_2 \longrightarrow \mathrm{Bicat}$

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[Romö, 24'] $\rightarrow$ $h_2 : CSS_2 \longrightarrow Bicat$

# WARMUP: MONOIDAL CATEGORIES AS FIBRATIONS

$$
\begin{tikzcd}
\mathrm{MonCat} \arrow[r, hook] \arrow[d, phantom, "\cong" rotate=90] & {[\Delta^{op}, \mathrm{Cat}]_{ps}} \arrow[d, phantom, "\cong" rotate=90] \\
??? \arrow[r, hook] & \mathrm{OpFib}/\Delta^{op}
\end{tikzcd}
$$

# WARMUP: MONOIDAL CATEGORIES AS FIBRATIONS

$$\begin{tikzcd} \mathrm{MonCat} \arrow[r, hook, "?"] & {[\Delta^{op}, \mathrm{Cat}]_{ps}} \arrow[d] \\ & \mathrm{OpFib}\ \Delta^{op} \end{tikzcd}$$

$\otimes : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$

$u : * \longrightarrow \mathcal{V}$

etc.

**Prop** A functor $[\Delta^{op}, \mathrm{Cat}]$ correspond to a diagram

$$\begin{tikzcd}
C_0 \arrow[r, "s_0" description] & C_1 \arrow[l, shift right=3, "d_0"'] \arrow[l, shift left=3, "d_1"] \arrow[r, shift left=2, "s_0" description] \arrow[r, shift right=2, "s_1" description] & C_2 \arrow[l, shift right=4, "d_0"'] \arrow[l, "d_1" description] \arrow[l, shift left=4, "d_2"] \arrow[r, shift left=3] \arrow[r] \arrow[r, shift right=3] & \cdots \arrow[l, shift right=4] \arrow[l, shift right=1.5] \arrow[l, shift left=1.5] \arrow[l, shift left=4]
\end{tikzcd}$$

such that

$$\begin{cases}
d_i d_j = d_{j-1} d_i, & i<j \\
s_i s_j = s_j s_{i-1}, & i>j \\
d_i s_j = \begin{cases} s_{j-1} d_i, & i<j \\ \mathrm{id}, & i=j \text{ or } j+1 \\ s_j d_{i-1}, & i>j+1 \end{cases}
\end{cases}$$

(Jardine, 91')

**Thm** A pseudo functor $[\Delta^{op}, Cat]$ correspond to a diagram

$$
\begin{tikzcd}
C_0 \arrow[r, "s_0"] & C_1 \arrow[l, shift right=2, "d_0"'] \arrow[l, shift left=2, "d_1"] \arrow[r, shift left, "s_0"] \arrow[r, shift right, "s_1"'] & C_2 \arrow[l, shift right=3, "d_0"'] \arrow[l, "d_1"] \arrow[l, shift left=3, "d_2"] & \cdots
\end{tikzcd}
$$

such that

$$
\begin{cases}
d_i d_j \simeq d_{j-1} d_i, & i<j \\
s_i s_j \simeq s_j s_{i-1}, & i>j \\
d_i s_j \simeq \begin{cases} s_{j-1} d_i, & i<j \\ id, & i=j \text{ or } j+1 \\ s_j d_{i-1}, & i>j+1 \end{cases}
\end{cases}
$$

+ coherence diagrams

Ex) $(V, \otimes, 1\!\!1) \in \mathrm{MonCat}$

$$
\begin{tikzcd}[column sep=huge]
\star \arrow[r, "1\!\!1"] & V \arrow[l, shift right=2] \arrow[l, shift right=4] \arrow[r, shift left=2, "id \times 1\!\!1"] \arrow[r, shift right=2, "1\!\!1 \times id"'] & V \times V \arrow[l, shift right=2, "\otimes"'] \arrow[l, shift right=4] \arrow[l, shift right=6] \arrow[r, shift right=2] \arrow[r, shift right=4] \arrow[r, shift right=6] & V \times V \times V \arrow[l, shift right=2, "\otimes \times 1"'] \arrow[l, shift right=4, "1 \times \otimes"'] \arrow[l, shift right=6] \arrow[l, shift right=8] \cdots
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & V \times V \arrow[dr, "d_1"] \arrow[d, phantom, "\Nearrow \alpha"] & \\
V \times V \times V \arrow[ur, "d_2"] \arrow[dr, "d_1"'] & {} & V \\
 & V \times V \arrow[ur, "d_1"'] &
\end{tikzcd}
$$

$(a, b, c)$

$(a \otimes b, c)$

$a, b \otimes c$

$(a \otimes b) \otimes c$

$\uparrow \alpha$

$a \otimes (b \otimes c)$

PENTAGON:

$$
\begin{tikzcd}
d_1 d_1 d_1 \arrow[rr] \arrow[d] & & d_1 d_2 d_1 \arrow[d] \\
d^1 d^1 d^2 \arrow[dr] & & d^1 d^2 d^3 \arrow[dl] \\
 & d^1 d^2 d^2 &
\end{tikzcd}
$$

✓

i.e. $\mathrm{MonCat} = [\Delta^{op}, \mathrm{Cat}]_{ps} \,\Big|\, \left\{ \begin{array}{l} F : \Delta^{op} \to \mathrm{Cat} \text{ s.t.} \\ F_0 \simeq * \\ F_n \xrightarrow{\sim} F_1 \times \dots \times \underbrace{F_1}_{\mathcal{V}} \end{array} \right.$

$$\begin{tikzcd}
\mathrm{MonCat} \arrow[rr] \arrow[d, "\simeq"'] & & {[\Delta^{op}, \mathrm{Cat}]} \arrow[d, dash, "\simeq"] \\
\left\{ \begin{array}{c} \mathcal{V}^{\otimes} \\ \downarrow \\ \Delta^{op} \end{array} \text{ s.t. } \begin{array}{l} \mathcal{V}^{\otimes}_0 \simeq * \\ \mathcal{V}^{\otimes} \simeq \mathcal{V}_1 \times \dots \times \mathcal{V}_1 \\ \text{(via certain lifts} \end{array} \right\} \arrow[rr, hook] & & \mathrm{OpFib}_{\Delta^{op}}
\end{tikzcd}$$

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$\mathcal{V}$

$$\begin{tikzcd}
\mathrm{MonCat} \arrow[r] \arrow[d, "\simeq"'] & {[\Delta^{op}, \mathrm{Cat}]} \arrow[d, "\simeq"] \\
\left\{ \begin{array}{c} \mathcal{V}^{\otimes} \\ \downarrow \\ \Delta^{op} \end{array} \text{ s.t. } \begin{array}{l} \mathcal{V}^{\otimes}_0 \simeq * \\ \mathcal{V}^{\otimes}_n \simeq \mathcal{V}_1 \times \dots \times \mathcal{V}_1 \end{array} \right\} \arrow[r, hook] & \mathrm{OpFib}/_{\Delta^{op}}
\end{tikzcd}$$

why take the trouble?

# WARMUP 2: monoidal homotopy 1-categories

Yoga: want to define monoidal $(\infty,1)$-categories as

$$\Delta^{op} \longrightarrow Cat_{(\infty,1)}$$

but these are hard (coherence all the way to $\infty$).

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Yoga: want to define monoidal $(\infty,1)$-categories as

$$\Delta^{op} \longrightarrow \mathrm{Cat}_{(\infty,1)}$$

but these are hard (coherence all the way to $\infty$).

Sol/ $[\Delta^{op}, \mathrm{Cat}] \rightsquigarrow \mathrm{Cocart}/\Delta^{op}$

coherence $\longrightarrow$ cartesianess

Def) A monoidal $\infty$-category is a cocartesian fibration

[DAG-III] $\mathcal{V}^{\otimes} \to \Delta^{op}$ s.t. $\mathcal{V}_0 \simeq *$

$\mathcal{V}_n \simeq \mathcal{V}_1 \times \cdots \times \mathcal{V}_1$

**Def)** A monoidal $\infty$-category is a cocartesian fibration

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**DESIDERATA:**

$$\begin{tikzcd} \Delta^{op} \arrow[r, "\mathcal{V}"] \arrow[rr, bend right, dashed, "h_1\mathcal{V}\ \triangle!"'] & \mathrm{Cat}_\infty \arrow[r, "h_1"] & \text{“}\mathrm{Cat}\text{”} \end{tikzcd}$$

$$\begin{tikzcd} h_1\mathcal{V}^{\otimes} \arrow[d] \\ \Delta^{op} \end{tikzcd}$$

your favourite model for $\infty$-categories

$$\Delta^{op} \xrightarrow{N} Cat_\infty \xrightarrow{h_1} \text{“}Cat\text{”}$$

$\subseteq$ quasicats in simplicial sets

a simplicial map

$\Delta$ the nerve of the (2,1)-category of categories

0-simplices: categories $\mathcal{C}, \mathcal{D}, \mathcal{E}, \ldots$

1-simplices: functors $F: \mathcal{C} \to \mathcal{D}$ $\longleftarrow$ $N_2 \underline{\mathcal{C}at}$

~~"$\mathcal{C}at$"~~

2-simplices: natural isos

$$\begin{tikzcd} & \mathcal{D} \arrow[dr, "G"] & \\ \mathcal{C} \arrow[ur, "F"] \arrow[rr, "H"'] \arrow[rr, phantom, shift left=4, "\Downarrow\eta"] & & \mathcal{E} \end{tikzcd}$$

the nerve of the (2,1)-category of categories

higher simplices <u>witness</u> the commutativity of pasting diagrams

e.g.

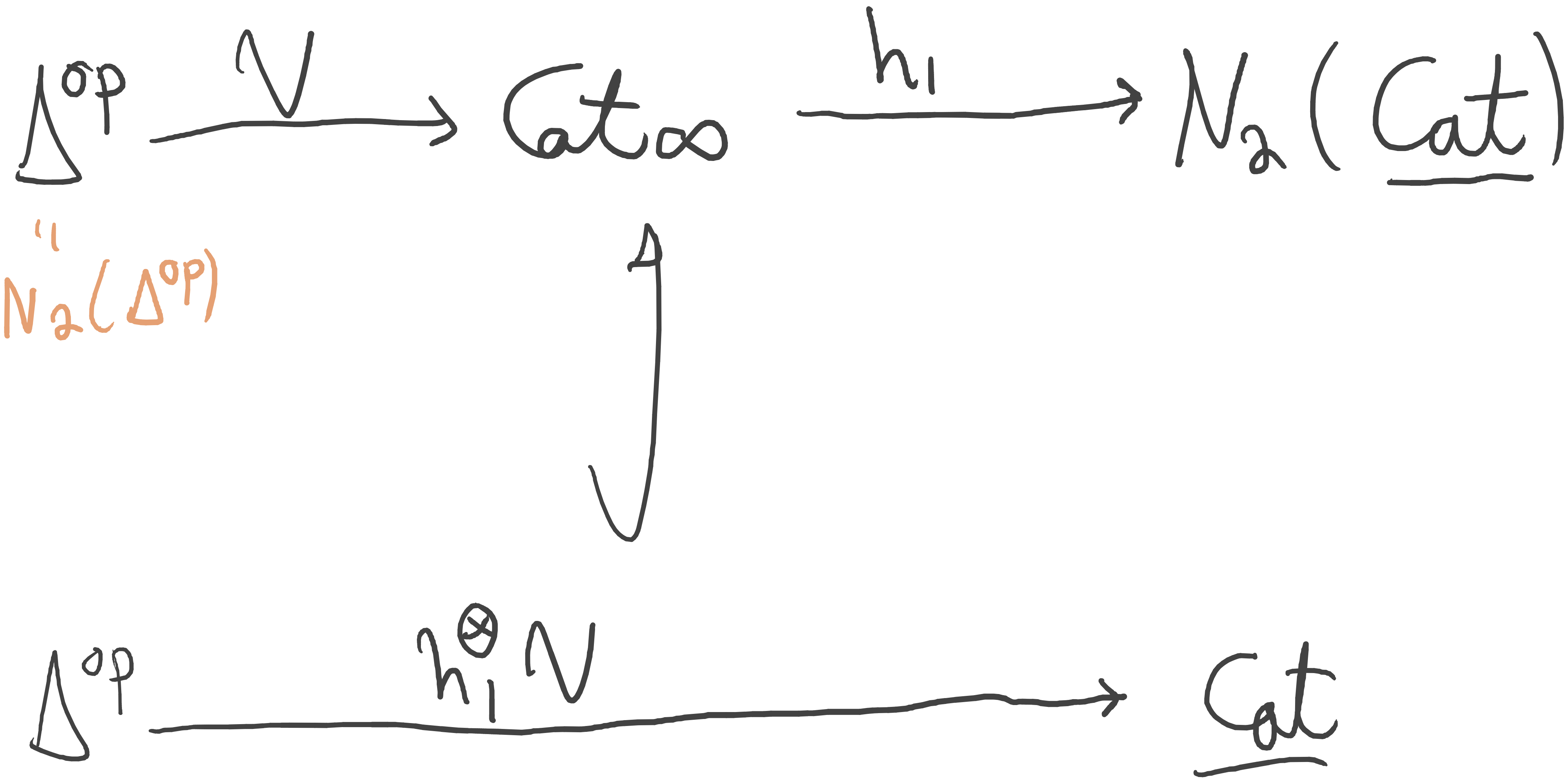
More generally:

$$\begin{tikzcd}
\mathrm{sSet} \arrow[dr, dashed, bend left=10] \arrow[dr, phantom, "\perp"] & \\
\Delta \arrow[u, "y"] \arrow[r] & \mathrm{Bicat} \arrow[ul, bend right=30]
\end{tikzcd}
\qquad N_2(B)_n = \mathrm{Ps}([n], B)$$

$$\mathrm{Bicat} \begin{cases} \text{invertible modifications} \\ \text{ps natural isos} \\ \text{pseudofunctors} \end{cases}$$

- $N_2$ is a right adjoint
- $N_2$ is fully faithful

- $N_2 : Cat_{(2,1)} \to sSet$ is fully faithful



$$
\begin{tikzcd}
\Delta^{op} \arrow[r, "V"] & Cat_\infty \arrow[r, "h_1"] & N_2(\underline{Cat}) \\
\Delta^{op} \arrow[rr, "h_1^{\otimes} V"] & & \underline{Cat}
\end{tikzcd}
$$

$\shortparallel$

$N_2(\Delta^{op})$

# MONOIDAL BICATEGORIES

$$\begin{array}{c} \mathrm{2Fib}/\Delta^{op} \\ \Big\| \wr \quad \text{[Baković '11]} \quad \text{[Buckley '13]} \\ \mathrm{MonBicat} \hookrightarrow [\Delta^{op}, \mathrm{Bicat}^{ps}]_{tri} \end{array}$$

# MONOIDAL BICATEGORIES

$$N^{\otimes} \to \Delta^{op}$$

$$\begin{tikzcd}
\text{fibred bicategories s.t. } N^{\otimes}_0 \simeq *,\ N^{\otimes}_n \simeq N^{\otimes}_1 \times \cdots \times N^{\otimes}_1 \arrow[r, hook] \arrow[d, phantom, "\wr"] & 2\mathrm{Fib}/\Delta^{op} \arrow[d, phantom, "\wr"] \\
\mathrm{MonBicat} \arrow[r, hook] & {[\Delta^{op}, \mathrm{Bicat}^{ps}]_{tri}}
\end{tikzcd}$$

takes care of the coherence!

$$* \longrightarrow B \;\overset{\otimes}{\leftleftarrows}\; B \times B \;\leftleftarrows\; B \times B \times B \cdots$$

**Def** A monoidal $(\infty,2)$-category is a simplicial map $\Delta^{op} \xrightarrow{\mathcal{V}} \mathrm{Cat}_{(\infty,2)}$ such that

$$\begin{cases} \mathcal{V}[0] \simeq * \\ \mathcal{V}[1] \times \cdots \times \mathcal{V}[1] \xrightarrow{\sim} \mathcal{V}[n] \end{cases}$$

**Def** A monoidal $(\infty,2)$-category is a simplicial map $\Delta^{op} \xrightarrow{\mathcal{V}} \mathrm{Cat}_{(\infty,2)}$

$\hookrightarrow \in \mathrm{Cat}_{(\infty,1)}$

such that $\begin{cases} \mathcal{V}[0] \simeq * \\ \mathcal{V}[1] \times \cdots \times \mathcal{V}[1] \xrightarrow{\sim} \mathcal{V}[n] \end{cases}$

i.e. a 2-coCartesian fibration

$$\begin{array}{c} \mathcal{V}^{\otimes} \\ \downarrow \\ \Delta^{op} \end{array} \quad \text{s.t.} \quad \begin{array}{l} \mathcal{V}^{\otimes}_0 \simeq * \\ \mathcal{V}_1 \times \cdots \times \mathcal{V}_1 \xrightarrow{\sim} \mathcal{V}_n \end{array}$$

So if we had $h_2 : \mathrm{Cat}_{(\infty,2)} \longrightarrow$ "Bicat"

$$\Delta^{op} \xrightarrow{N^{\otimes}} \mathrm{Cat}_{(\infty,2)} \xrightarrow{h_2} N_3(\mathrm{Bicat})$$

$$\Delta^{op} \xrightarrow{h_2^{\otimes} N^{\otimes}} \mathrm{Bicat}$$

Thm/def (Rómo, 24') $h_2 : CSS_2 \longrightarrow Bicat$

complete 2-fold Segal spaces model $(\infty,2)$-categories

$$\Delta^{op} \xrightarrow{B} CSS_2 \xrightarrow{\hat{h}_2} Bicat \begin{cases} \text{invertible modifications} \\ \text{pseudonatural isos} \\ \text{pseudofunctors} \\ \text{bicats} \end{cases}$$

$$\begin{tikzcd}
\Delta^{op} \arrow[r, "B"] & CSS_2 \arrow[r, "h_2"] & \text{"}N_3\text{"}(Bicat) \arrow[d] \\
\Delta^{op} \arrow[rr, "h_2^{\otimes}B"] & & Bicat
\end{tikzcd}$$

$\left\{\begin{array}{l}\text{invertible modifications} \\ \text{pseudonatural isos} \\ \text{pseudofunctors} \\ \text{bicats}\end{array}\right.$

Cor A monoidal $(\infty,2)$-category $\begin{array}{c} B^{\otimes} \\ \downarrow \\ \Delta^{op} \end{array}$ defines a monoidal bicategory $\begin{array}{c} h_2B^{\otimes} \\ \downarrow \\ \Delta^{op} \end{array}$ under the construction above.

# SYMMETRIC HOMOTOPY BICATEGORIES

Do everything again with $\mathrm{Fin}_*$ replacing $\Delta^{op}$:

**Def** A symmetric monoidal $(\infty, 2)$-category is a cocartesian fibration $V^{\otimes} \to \mathrm{Fin}_*$ s.t. $\begin{cases} V_{*} \simeq * \\ V_n \xrightarrow{\sim} V_1 \times \cdots \times V_1 \end{cases}$

e.g.
$$\begin{tikzcd} \{*, \underline{0}, \underline{1}\} \arrow[d, "\text{swap}"'] \arrow[dr] & \\ \{*, \underline{0}, \underline{1}\} \arrow[r] & \{*, \underline{0}\} \end{tikzcd} \subseteq \mathrm{Fin}_*$$

lifts to

$$\begin{tikzcd} C_1 \times C_1 \arrow[d, "\text{swap}"'] \arrow[dr, "\otimes"] \arrow[d, phantom, shift left=6, "\beta \Downarrow"] & \\ C_1 \times C_1 \arrow[r, "\otimes"'] & C_1 \end{tikzcd}$$

(the braiding!)

Same workflow: $\mathrm{Fin}_* \xrightarrow{\mathcal{V}} CSS_2 \xrightarrow{h_2} N_3(\mathrm{Bicat})$

$\exists$ f.f.

$\mathrm{Fin}_* \xrightarrow{h_2 \mathcal{V}} \mathrm{Bicat}$

**Def** The symmetric monoidal homotopy bicategory of an SM $(\infty,2)$-category $\mathcal{V}^{\otimes} \to \mathrm{Fin}_*$ is $h_2\mathcal{V}^{\otimes} \to \mathrm{Fin}_*$

**Def/cor)** The underlying monoidal category to a symmetric monoidal category $\mathcal{V}^{\otimes} \to \mathrm{Fin}_*$ is obtained by pre-composing w/ $\Delta^{op} \hookrightarrow \mathrm{Fin}_*$.

**Def** The space of symmetric enhancements to a monoidal category $\mathcal{V}^{\otimes} \to \Delta^{op}$ is the space of lifts

$$
\begin{tikzcd}
 & \Delta^{op} \arrow[d, hook] \\
\mathcal{V}^{\otimes} \arrow[ur, dashed] \arrow[r, "+\text{conditions}"'] & \mathrm{Fin}_*
\end{tikzcd}
\quad \subseteq \; \mathrm{sSet}_{/\mathrm{Fin}_*}(\mathcal{V}^{\otimes}, \Delta^{op})
$$

Summary: given a (symmetric) monoidal $\mathcal{V}^{\otimes} \to \Delta^{op}$ or $\mathrm{Fin}_*$ we construct $h_2\mathcal{V}^{\otimes} \to \Delta^{op}$ or $\mathrm{Fin}_*$ by enhancing $h_2 : \mathrm{Cat}_{(\infty,2)} \longrightarrow \mathrm{Bicat}$.

Leftovers/conjectures:

- find a base for braided/etc. categories
- find a base/characterize fully valizable bicategories
- in general, characterize the "Segalification" of a free structured category

MERCI