A MODEL STRUCTURE FOR 2-FIBRATIONS

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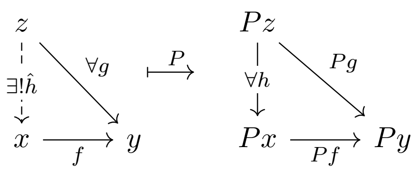
Abstract. Notes for a talk at the ATCAT about a tentative formulation of the unstraightening of 2-categories via model categories. Based on work in progress with C. Bardomiano, M. Sarazola, J. Nickel, S. Toro Oquendo, and P. Verdugo.

1. Review of cartesian fibrations

1.1. **Definitions.** A **cartesian fibration**[1] is a bundle of categories with a functorial path lifting property between the fibers. We will spell the precise definition and then explain this intuition.

Definition 1.1.1. *Given a functor $P : \mathcal{E} \to \mathcal{B}$, a morphism $f : x \to y$ in $\mathcal{E}$ is **P-cartesian** if for all $g : z \to y$ and $h : Pz \to Px$ such that $Pf.h = Pg$, there exists a unique $\hat{h} : z \to x$ lifting h and such that $fh = g$. Diagramatically:*



The point is that P-cartesian lifts of morphisms in $\mathcal{B}$ capture the notion of functorial lifts.

Definition 1.1.2. *A functor $P : \mathcal{E} \to \mathcal{B}$ is a **cartesian fibration** or **category fibred over** $\mathcal{B}$ if every morphism $f : x \to Py$ has a cartesian lift.*

Remark 1.1.3. The existence of lifts for morphisms of the form $f : x \to Py$ can be captured by a lifting property (against $[0] \xrightarrow{1} [1]$), but the notion of cartesianness cannot. We will get back to this in Section 1.3.

Example 1.1.4. If $\mathcal{C}$ is any category, then the taking domains defines the *domain fibration* dom : $\mathcal{C}^[1] \to \mathcal{C}$. If $\mathcal{C}$ has pullbacks, then taking codomains defines the *codomain fibration*

Example 1.1.5. Let VB be denote the category of continuous vector bundles, and $P : \mathrm{VB} \to \mathrm{Top}$ be the functor sending a bundle to its base. Then a morphism in VB is P-cartesian iff it defines a pullback square, whence P is a cartesian fibration, since pullback bundles always exist.

Example 1.1.6. The category of elements of a presheaf $F : \mathcal{C} \to \mathrm{Set}$ is a fibration with the property that the cartesian lifts are unique. Such fibrations are called *discrete*.

Example 1.1.7. If $\mathcal{E} \to X$ is a category fibred over a set, then $\mathcal{E}$ is a groupoid. An example is the isomorphism class functor $\mathcal{G} \to h_0(\mathcal{G})$.

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[1]In the literature these are usually called "Grothendieck fibrations", with the term cartesian being reserved for fibrations of $(\infty, 1)$-categories. Since the concepts are compatible (Theorem Theorem 1.3.5), we see no need for this distinction (and there are already too many things named after Grothendieck).

The existence of cartesian lifts can be interpreted as a unique path lifting property. Given a "path" $f : x \to Py$ in the base, we can lift it to a "path" $\hat{f} : \hat{x} \to y$. This lift is unique: any other lift $\widetilde{f} : \widetilde{x} \to y$ is canonically isomorphic to $\hat{tilde}$ (in $\hat{f} \cong \widetilde{f}$ in $\mathcal{E}/y$):

$$\begin{array}{ccc} \hat{x} & \xrightarrow{\hat{f}} & y \\ \downarrow & & \| \\ \widetilde{x} & \xrightarrow[\widetilde{f}]{} & y \end{array} \qquad \xrightarrow{P} \qquad \begin{array}{ccc} x & \xrightarrow{f} & Py \\ \| & & \| \\ x & \xrightarrow[f]{} & Py \end{array}$$

1.2. **As pseudofunctors to** Cat. If $P : \mathcal{E} \to \mathcal{B}$ is a cartesian fibration, then taking fibers defines a pseudofunctor $\mathcal{B} \to \mathrm{Cat}$. In fact, this construction assembles into an equivalence of bicategories.

Let $\mathrm{Fib}(\mathcal{B})$ be the bicategory of cartesian fibrations over $\mathcal{B}$, functors which preserve cartesian morphisms, and natural transformations between them. Let $[\mathcal{B}^{\mathrm{op}}, \mathrm{Cat}]_{ps}$ be the bicategory of pseudofunctors, pseudonatural transformations, and modifications.

Theorem 1.2.1. *Taking fibers defines a biequivalence* $\mathcal{T}_{\mathcal{B}} : \mathrm{Fib} \to [\mathcal{B}^{\mathrm{op}}, \mathrm{Cat}]_{ps}$.

The inverse of $\mathcal{T}_{\mathcal{B}}$ is the *Grothendieck* or unstraightening construction, which assembles a pseudofunctor $F : \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}$ as a fibration $\int_{\mathcal{B}} F \to \mathcal{B}$ such that the fiber over $x \in \mathcal{B}$ is $F(x)$, and pseudofunctoriality is recorded by cartesian lifts.

Example 1.2.2. The domain fibration from Example 1.1.4 straightens the functor $\mathcal{C} \to \mathrm{Cat}$ sending $c \in \mathcal{C}$ to the undercategory $c/\mathcal{C}$, and the codomain fibration straightens the pseudofunctor sending $c \in \mathcal{C}$ to the overcategory $\mathcal{C}/c$.

Example 1.2.3. The projection fibration from Example 1.1.5 straightens the functor $\mathrm{VB}(-) : \mathrm{Top}^{\mathrm{op}} \to \mathrm{Cat}$ sending a space X to the category of vector bundles over X.

Example 1.2.4. A discrete fibration is the straightening of a category fibred in sets.

Example 1.2.5. The fibration $h_0 : \mathcal{G} \to h_0(\mathcal{G})$ from Example 1.1.7 straightens the functor sending an object in $\mathcal{G}$ to its connected component.

1.3. **via model categories.** The equivalence in the previous section matches the familiar concept of pseudofunctors with the *ad hoc* definition of cartesian fibration. Moser-Sarazola recently showed that this can also be achieved by a Quillen equivalence, and that this construction is compatible with the straightening-unstraightening of $(\infty, 1)$-categories.

The idea of their proof is to find a model structure on the slice $\mathrm{Cat}/\mathcal{C}$ encoding cartesian fibrations as fibrant objects. This means, in particular, capturing their defining properties as lifting properties. However, this doesn't quite work out for cartesian morphisms, which are "distinguished" morphisms in a category.

Definition 1.3.1. *A* ***marked category*** *is a category* $\mathcal{C}$ *with a distinguished set of morphisms* E*.*

Let Cat^+ be the category of marked categories and functors preserving the marking. Let $\mathcal{C}^\sharp$ denote a category $\mathcal{C}$ with all morphisms marked, and call a marked functor $P : \mathcal{E}^+ \to \mathcal{C}^\sharp$ a *cart-marked fibration* if it is a fibration and the marked morphisms of $\mathcal{E}^+$ are the P-cartesian morphisms.

Proposition 1.3.2. *A marked functor* $\mathcal{E}^+ \to \mathcal{C}^\sharp$ *is a cart-marked fibration iff it has the RLP against the morphisms in* $\mathrm{Cat}^+/\mathcal{C}^\sharp$*:*

(1) *the inclusion* $[0] \xrightarrow{1} [1]^\sharp \to \mathcal{C}^\sharp$,
(2) *the inclusion* $(\Lambda^2[2], \{1 \to 2\}) \to ([2], \{1 \to 2\}) \to \mathcal{C}^\sharp$,
(3) *the functor* $([2] \sqcup_{\Lambda^2[2]} [2], \{1 \to 2\}) \to ([2], \{1 \to 2\}) \to \mathcal{C}^\sharp$,
(4) *the inclusion* $\mathbb{I}^\flat \to \mathbb{I}^\sharp \to \mathcal{C}^\sharp$,
(5) *the inclusion* $([2], \{0 \to 1, 1 \to 2\}) \to [2]^\sharp \to \mathcal{C}^\sharp$,

From this, the idea of their proof is to use a recent theorem of Guetta-Moser-Sarazola-Verdugo which constructs a cofibrantly generated model category from specified fibrant objects, cofibrations, and fibrations between fibrant objects.

Theorem 1.3.3. *There exists a model structure on* $\mathrm{Cat}^+_{/\mathcal{C}^\sharp}$ *whose*

1. *the fibrant objects are the cart-marked fibrations.*
2. *the cofibrations are the injective-on-objects functors.*
3. *the weak equivalences between fibrant objects are the equivalences of categories,*
4. *the fibrations between fibrant objects are the isofibrations.*

Consider now the projective model structure on $[\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}]_{\mathrm{proj}}$, that is the one with weak equivalences and fibrations defined levelwise as equivalences of categories and isofibrations (the weak equivalences and fibrations of Cat).

Theorem 1.3.4. *There exists a Quillen equivalence*

$$\mathcal{T}^+_{\mathcal{C}} : \mathrm{Cat}^+_{/\mathcal{C}^\sharp} \to [\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}]_{proj} : \int_{\mathcal{C}}^{+}$$

The idea of using marked categories to encode cartesian fibrations is not new. In fact, a huge chunk of Joyal-Lurie's theory of $(\infty, 1)$-categories stems from their straightening-unstraightening construction, which is formalized by marked simplicial sets.

The following proposition asserts the agreement between the model structure on cartesian fibrations of $(\infty, 1)$-categories and ordinary categories:

Theorem 1.3.5. *The following diagram of right Quillen functors commutes up to weak equivalence.*

$$\begin{array}{ccc} [\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}^+]_{proj} & \xrightarrow{\sim} & \mathrm{Cat}^+_{/\mathcal{C}^\sharp} \\ \downarrow & & \downarrow \\ [\mathcal{C}^{\mathrm{op}}, \mathrm{sSet}^+]_{proj} & \xrightarrow{\sim} & \mathrm{sSet}^+_{/N\mathcal{C}^\sharp} \end{array}$$

2. A model structure for 2-fibrations

2.1. **2-fibrations.** Fibrations of bicategories, or *2-fibrations*, are defined *ad hoc* as the result of an straightening construction for trifunctors from a bicategory to the tricategory Bicat_{ps}. The correct definition of 2-fibration in this case is due to Buckley, building on preliminary work of Baković and Hermida which analyzed particular cases (such as 2-categories instead of bicategories).

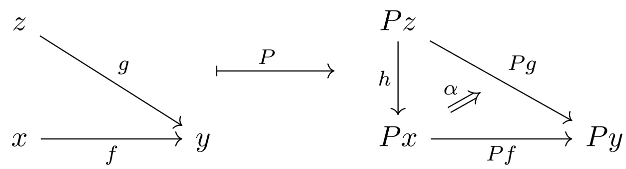
Nevertheless, we restrict ourselves to 2-fibrations between 2-categories. This is justified because our goal is to formulate an straightening-unstraightening for 2-fibrations in terms of Quillen equivalences, and:

- strict functors suffice because the model structure overcomes the need for weak inverses
- strict 2-categories suffice because the inclusion $2\mathrm{Cat}_{st} \hookrightarrow \mathrm{Bicat}_{st}$ turns out to be a Quillen equivalence.

These two observations are due to Lack.

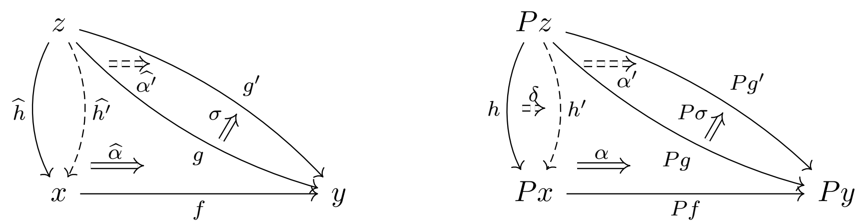
Definition 2.1.1. *Given a 2-functor* $P : \mathcal{E} \to \mathcal{B}$*, a 1-cell* $f : x \to y$ *in* $\mathcal{E}$ *is* ***P-cartesian*** *if*

1. *For all* $g : z \to y$*,* $h : Pz \to Px$*, and invertible 2-cell* $\alpha : Pf.h \Rightarrow Pg$*, there exists a morphism* $\widehat{h} : z \to x$ *and invertible 2-cells* $\widehat{\alpha} : f\widehat{h} \Rightarrow g$ *and* $\widehat{\beta} : P\widehat{h} \Rightarrow h$ *such that* $\alpha.(1_{Pf} * \widehat{\beta}) = P\widehat{\alpha}$*. Diagramatically:*



Then

(2) *For all* $\sigma\colon g \Rightarrow g'$, $h, h' : Pz \to Px$, *and invertible 2-cells* $\alpha\colon Pf.h \Rightarrow Pg$ *and* $\alpha'\colon Pf.h' \Rightarrow Pg'$, *choose lifts* $(\widehat{h}, \widehat{\alpha}, \widehat{\beta})$ *and* $(\widehat{h'}, \widehat{\alpha'}, \widehat{\beta'})$. *Then, for any* $\delta\colon h \Rightarrow h'$ *in* $\mathcal{C}$ *with* $\alpha'.Pf\delta = P\sigma.\alpha$ *there exists a unique* $\widehat{\delta}\colon \widehat{h} \Rightarrow \widehat{h'}$ *such that* $\widehat{\alpha'}.f\widehat{\delta} = \sigma.\widehat{\alpha}$ *and* $\delta.\widehat{\beta} = \widehat{\beta'}.P\widehat{\delta}$. *Diagramatically:*



Definition 2.1.2. *A* ***2-fibration*** *is a 2-functor* $P : \mathcal{E} \to \mathcal{B}$ *such that*

(1) *every 1-cell of the form* $f : x \to Py$ *has a cartesian lift.*
(2) P *is locally a cartesian fibration.*
(3) *the horizontal composition of* P*-cartesian 1-cells is* P*-cartesian.*

Remark 2.1.3. There are

Example 2.1.4. Cartesian fibrations are 2-fibrations.

Example 2.1.5 (Hermida)**.** The codomain functor Fib $\to$ Cat is a 2-fibration.

Example 2.1.6. If $\mathcal{E} \to \mathcal{C}$ is a fibration over a 1-category, then $\mathcal{E}$ is a (2,1)-category, and a 2-groupoid if $\mathcal{C}$ is a groupoid. An example is the 2-functor $\mathcal{G} \to h_1(\mathcal{G})$ which takes isomorphism classes of 1-cells. The sequence

$$\mathcal{G} = h_2\mathcal{G} \to h_1\mathcal{G} \to h_0\mathcal{G}$$

reminisces the Postnikov tower of a 2-type.

The main utility of 2-fibrations is to handle trifunctors to 2Cat in terms of strict 2-functors with special properties. This is justified by the following result, where the left hand side has the projective model structure induced by the Lack model structure on 2Cat.

Theorem 2.1.7 (Buckley)**.** *There exists an equivalence of tricategories*

$$\mathcal{T}_{\mathcal{B}} : [\mathcal{B}^{\text{op}}, 2\text{Cat}]_{proj} \to 2\text{Fib}/\mathcal{B}.$$

Example 2.1.8. Example 1.1.5 can be enhanced to a 2-fibration by consider the 2-functor $\mathcal{V}B(-) : \mathcal{T}\text{op}^{\text{op}} \to 2\text{Cat}$, where Top now is the naive homtopy 2-category of spaces and $\mathcal{V}B(X)$ refines $VB(X)$ by allowing homotopies between vector bundles.

Example 2.1.9. Taking under-2-categories defines a 2-functor $\mathcal{B}^{\text{op}} \to 2\text{Cat}$, and so do over-2-categories if $\mathcal{B}$ has pullbacks. By straightening these functors we obtain domain and codomain fibrations generalizing those of Example 1.1.4.

Example 2.1.10. The association $\mathcal{V} \to \mathcal{V}\text{Cat}$ defines a 2-functor $\text{MonCat}_{\text{lax}} \to 2\text{Cat}$ which can be straightened into a 2-fibration $\mathcal{E} \to \text{MonCat}_{\text{lax}}^{\text{op}}$. The fiber of a monoidal category $\mathcal{V}$ consists of the $\mathcal{V}$-categories.

2.2. **A model structure.**

Definition 2.2.1. *A **marked 2-category** is a 2-category with distinguished sets of 1- and 2-cells.*

Let $2\mathrm{Cat}^+$ be the category of marked 2-categories and functors preserving the marking. Let $\mathcal{B}^\sharp$ denote a 2-category with all morphisms marked, and call a marked functor $P : \mathcal{E}^+ \to \mathcal{C}^\sharp$ a *cart-marked 2-fibration* if it is a 2-fibration and the marked 1- and 2-cells of $\mathcal{E}^+$ are the P-cartesian 1- and 2-cells.

Proposition 2.2.2. *There exists a set of marked 2-functors such that a marked 2-functor $P : \mathcal{E}^+ \to \mathcal{C}^\sharp$ is cart-marked iff it has the RLP against the elements of this set.*

Sketch of proof. Local fibredness can be described by suspending the morphisms in Proposition 1.3.2, and adapting those allow for a description of cartesian lifts. One last marked 2-functor indicates that the composition of cartesian 1-cells is closed under horizontal composition. □

Theorem 2.2.3. *There exists a model structure on $2\mathrm{Cat}^+_{/\mathcal{B}^\sharp}$ whose*

1. *fibrant objects are the cart-marked 2-fibrations.*
2. *cofibrations are the Lack cofibrations.*
3. *the weak equivalences between fibrant objects are the biequivalences of 2-categories.*
4. *the fibrations between fibrant objects are the Lack fibrations.*

2.3. **Interlude: localizing 2-categories.** The Lack model structure on 2Cat induces a projective model structure on $[\mathcal{B}^op, 2\mathrm{Cat}]_{\mathrm{proj}}$. In order to check correctness of this model structure, we expect there to be a Quillen equivalence

$$\mathcal{T}_{\mathcal{B}} : 2\mathrm{Cat}^+_{/\mathcal{B}^\sharp} \leftrightarrows [\mathcal{B}^op, 2\mathrm{Cat}]_{\mathrm{proj}} : \int_{\mathcal{B}}^{+}$$

In mimicking the Moser-Sarazola proof, a technical issue arises. Namely, their construction of the left adjoint $\mathcal{T}_{\mathcal{B}}$ requires arbitrary localizations of morphisms in a 1-category, hence our analogous statement requires the arbitrary localization of 2-categories, in the sense of Bustillo-Pronk-Szyld.

Definition 2.3.1. *Let $\mathcal{B}$ and with a distinguished set of 1-cells $\mathcal{U}$. A **localization** of $\mathcal{B}$ at $\mathcal{U}$ is a 2-category $\mathcal{B}[\mathcal{U}^{-1}]$ equipped with a 2-functor $\gamma : \mathcal{B} \to \mathcal{B}[\mathcal{U}^{-1}]$ such that:*

1. *γ sends elements of $\mathcal{U}$ to equivalences.*
2. *For each 2-category $\mathcal{C}$, the induced map*

$$\gamma^* : \mathit{2Func}(\mathcal{B}\mathcal{U}^{-1}], \mathcal{C}) \to \mathit{2Func}_{\mathcal{U}}(\mathcal{B}, \mathcal{C})$$

is an equivalence of categories, where $\mathit{2Func}_{\mathcal{U}}(\mathcal{B}, \mathcal{C})$ are the 2-functors that send elements of $\mathcal{U}$ to equivalences in $\mathcal{D}$.

Proposition 2.3.2. *The localization of a small 2-category at 1-cells always exists.*

Sketch of proof. Use cocompleteness of 2Cat to glue an equivalence to each distinguished 1-cell. □

Definition 2.3.3. *Let $\mathcal{B}$ and with a distinguished set of 2-cells $\mathcal{U}$. A **localization** of $\mathcal{B}$ at $\mathcal{U}$ is a 2-category $\mathcal{C}[\mathcal{V}^{-1}]$ equipped with a 2-functor $L : \mathcal{C} \to \mathcal{C}[\mathcal{V}^{-1}]$ such that:*

1. *L sends elements of $\mathcal{W}$ to equivalences.*
2. *For each 2-category $\mathcal{D}$, the induced map*

$$L^* : Hom(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D}) \to Hom^{\mathcal{W}}(\mathcal{C}, \mathcal{D})$$

is an equivalence of categories, $Hom^{\mathcal{W}}(\mathcal{C}, \mathcal{D})$ are the 2-functors that send elements of $\mathcal{W}$ to equivalences in $\mathcal{D}$.

Proposition 2.3.4. *The localization of a small 2-category at 2-cells always exists.*

Sketch of proof. Localize locally. Since localization preserves products, there is a canonical way to extend this to the whole 2-category. □

Corollary 2.3.5. *A localization of a small 2-category in the sense of Bustillo-Pronk-Szyld always exists. It can be obtained by first localizing the 2-cells, then the 1-cells, as above.*

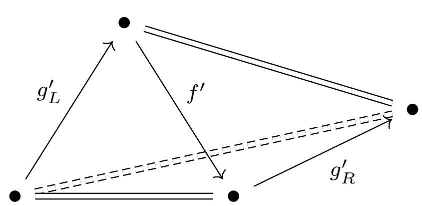
Nevertheless, this construction does not guarantee (strict) functoriality of the localization (in fact I suspect it is not). Still, a localization satisfying a weaker universal property exists functorially:

Proposition 2.3.6. *There exists a functor* $2\mathrm{Cat}^+ \to 2\mathrm{Cat}$ *which strictly inverts the marked 1- and 2-cells.*

Proof. This functor is the right adjoint to the functor $2\mathrm{Cat} \to 2\mathrm{Cat}^+$ marking invertible 1- and 2-cells. The fact that the latter is a left adjoint follows from the adjoint functor theorem for locally presentable categories by checking explicitly the preservation of products, equalizers, and filtered colimits of 2-categories. □

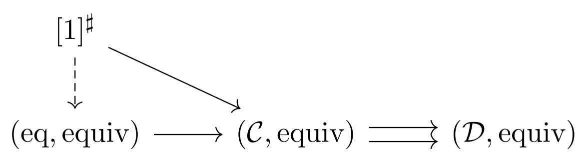
Remark 2.3.7. The 2-functor $2\mathrm{Cat} \to 2\mathrm{Cat}^+$ which marks equivalences is **not** a right adjoint. Let:

- $\mathcal{C} = (f, g, \text{etc.})$ be the walking adjoint equivalence.
- $\mathcal{D} = (f', g'_L, g'_R, \text{etc.})$ be the walking "ambidextrous" equivalence (this should be spelled out somewhere, for now see the picture below).



Let $L, R : \mathcal{C} \rightrightarrows \mathcal{D}$ be the inclusions sending g to g'_L and g'_R, respectively. Their equalizer is the single arrow f in $\mathcal{C}$, which has no equivalences.

Consider the category $[1]^\sharp$ with a single marked arrow. The $f : [1]^\sharp \to (\mathcal{C}, \text{equiv})$ picking up f is such that $Lf = Rf$, so there is an induced functor $[1]^\sharp \to \text{eq}$.



The dashed arrow does not preserve the marking.

From this I believe we can achieve the result in the beginning of the section. The right adjoint is obtained by modifying Buckley's Grothendieck construction.

2.4. **Comparison with $(\infty, 2)$-fibrations.** García-Stern recently developed the notion of fibred $(\infty, 2)$-categories, i.e. the $\infty-$ counterpart of 2-fibrations.

Hence we expect a weakly commuting diagram of Quillen adjunctions

$$\begin{array}{ccc} [\mathcal{B}^{\mathrm{op}}, 2\mathrm{Cat}^+]_{\mathrm{proj}} & \xrightarrow{\sim} & 2\mathrm{Cat}^+_{/\mathcal{B}^\sharp} \\ \downarrow & & \downarrow \\ [\mathcal{C}^{\mathrm{op}}, \mathrm{sSet}^\natural]_{\mathrm{proj}} & \xrightarrow{\sim} & \mathrm{sSet}^\natural_{/N\mathcal{B}^\sharp} \end{array},$$

where $\mathrm{sSet}^\natural$ is the category of marked-scaled simplicial sets (following Lurie's ∞-bicategories).

We expect this result to factor through the two-step localizations of the previous sections.