Eckmann-Hilton arguments

$a(bc) = (ab)c$

Def) A monoid is a set X w/ an associative map $\cdot : X \times X \to X$ w/ a unit $u \in X$, $\rightsquigarrow ua = a = au$

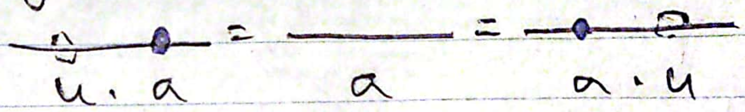
We can depict the elements of X by points, and the monoidal structure allow us to put them in a line

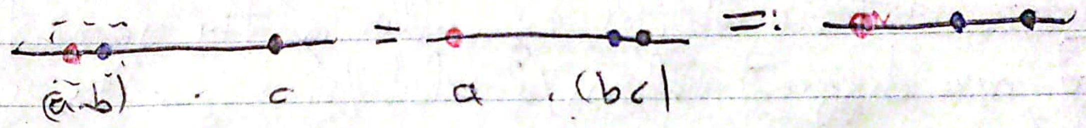
$=: \mu(a, b), \; a, b \in X$
$\in X$

Associativity + unitality ensure that these pictures are well defined.

$\hookrightarrow$ depict $u \in X$ by no dots: u = —— (instead of ——)

unitality =



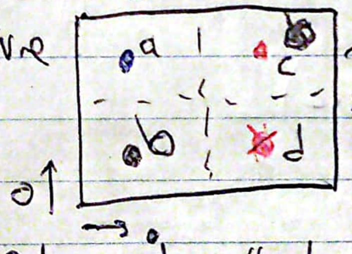


A monoid is abelian if

. Note that this is false in general: the points are not transparent to each other.

Suppose that $(X, \cdot, u)$ has another monoidal structure $(X, \circ, u_\circ)$, compatible w/ $\cdot$ in that $(a \cdot b) \circ (c \cdot d) = (a \circ c) \cdot (b \circ d)$.

Graphically, we can read this equation by saying that the picture



is well defined
$\overset{?}{=} (a \cdot b) \circ (c \cdot d)$ (vertical first)
$\overset{?}{=} (a \circ c) \cdot (b \circ d)$ (horizontal first)

This could be another unit in principle, but the axiom would imply $u_\cdot = u_\circ$ actually.

i.e. the 2nd multiplication "extruded" a new dimension to the pictures.

$1 \cdot 1 = 1 = u$ $\qquad$ $= (b \cdot 1) \circ (1 \cdot a) = (b \circ 1) \cdot (1 \circ a) = b \cdot a$

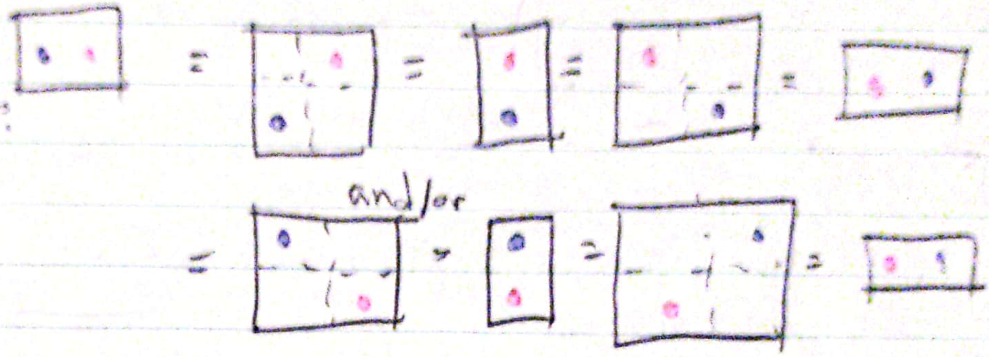
<u>Thm</u> (Eckmann-Hilton II) $\bullet = \circ$, and this commutes

<u>Proof</u> In equations $a \bullet b = (1 \circ a) \bullet (b \circ 1) = (1 \bullet b) \circ (a \bullet 1) = b \circ a$

and/or

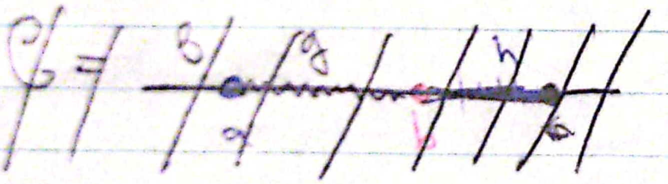
$= (a \circ 1) \bullet (1 \circ b) = (a \bullet 1) \circ (1 \bullet b) = a \circ b$

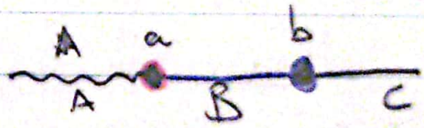
In pictures:



□

A <u>category</u> is what one obtains by allowing the segments of the monoid to have different colours/labels (w/ similar associative + unital properties)

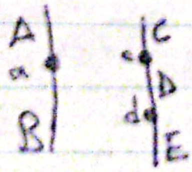


$C = $  $=: b \circ a$

$a : A \to B$, $\; b : B \to C$ $\; ; \; b \circ a : A \to C$

(Note that we can't ask for C to be commutative.)

A <u>monoidal category</u> is what we get by allowing to put the strings side by side

$\bullet \mapsto \otimes$  $=: a \otimes (d \circ c)$

Suppose ~~that~~ that $(C, \otimes)$ is a monoidal category w/ another monoidal structure $(C, \boxplus)$ compatible w/ $\otimes$ in the sense that there is a <u>natural iso</u> ↪ instead of equality

$$(a \otimes b) \boxplus (c \otimes d) \cong (a \boxplus c) \otimes (b \boxplus d)$$

$\simeq (b\otimes 1)\boxplus(1\otimes a) \simeq (b\boxplus 1)\otimes(1\boxplus a) \simeq b\otimes a$

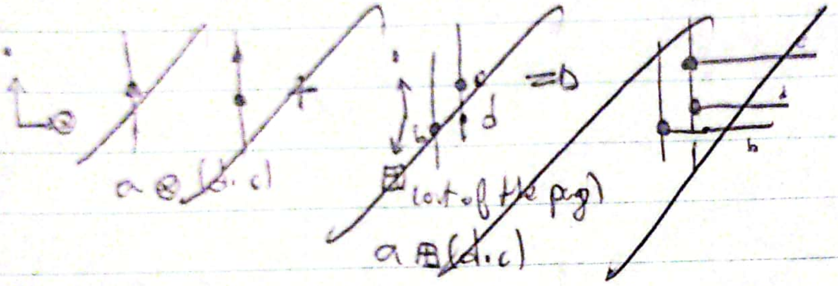
$\boxed{\text{Thm}}$ (Eckmann-Hilton II) $\otimes \simeq \boxplus$, and there's a natural iso $\beta_{x,y}: x\otimes y \to y\otimes x$

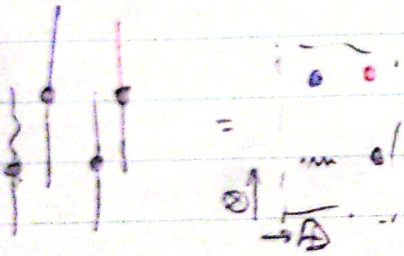
$\boxed{\text{Proof}}$ In eqs, $a\otimes b \simeq (1\boxplus a)\otimes(b\boxplus 1) \simeq (1\otimes b)\boxplus(a\otimes 1) \simeq b\boxplus a$

and/or

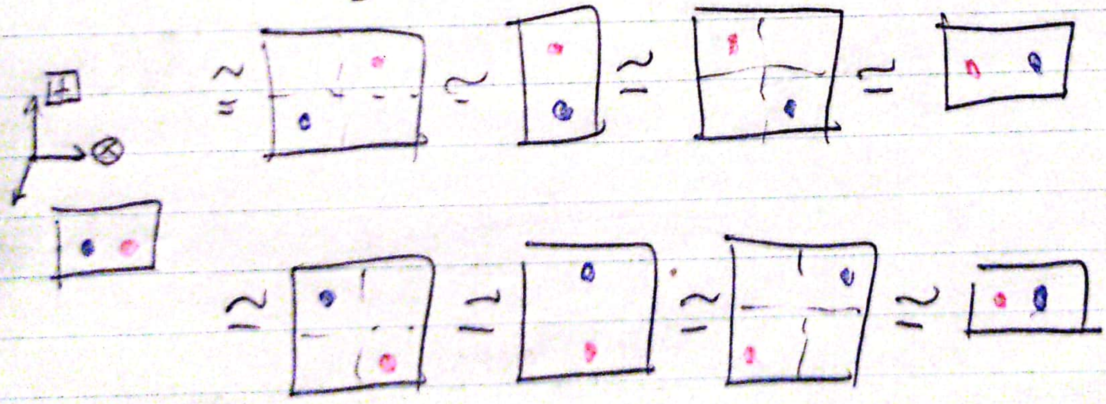
$\simeq (a\boxplus 1)\otimes(1\boxplus b) \simeq (a\otimes 1)\boxplus(b\otimes 1) \simeq a\boxplus b$

To make a picture proof, imagine that $\boxplus$ gives $(C,\otimes)$ a new direction of composition, which can be visualized by extruding  into strings 

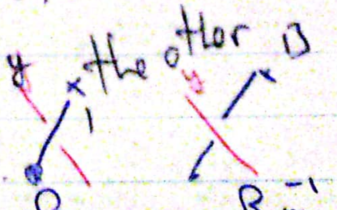




Then we have the same picture proofs

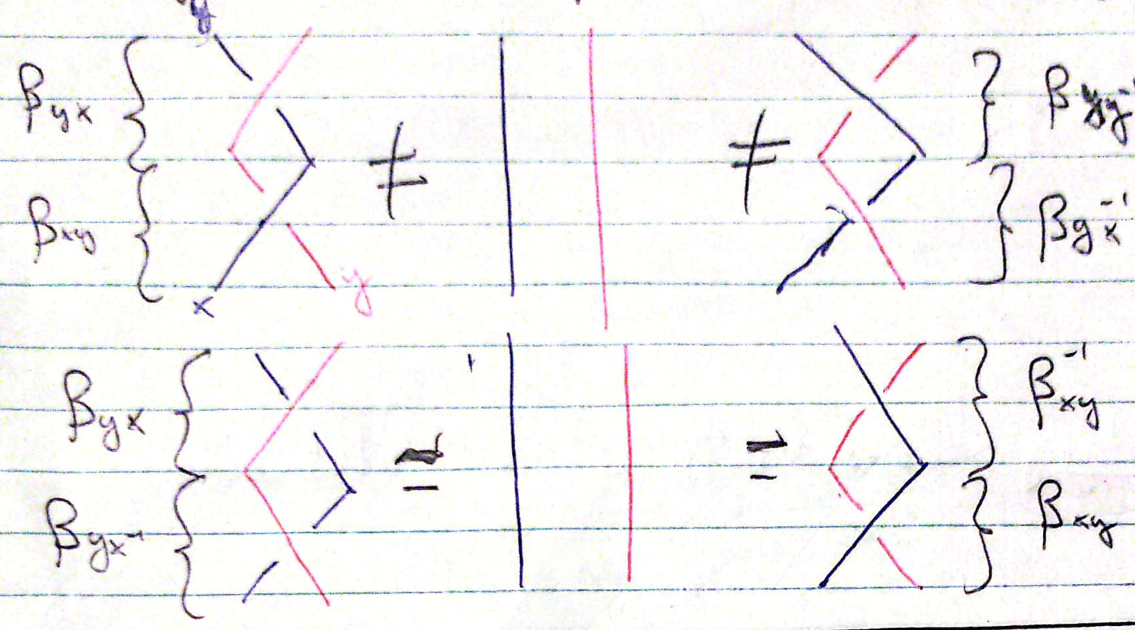


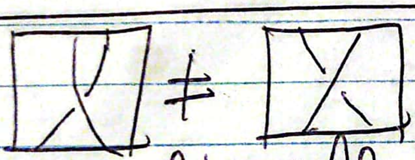
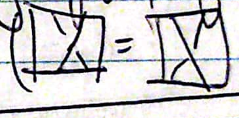
(picture this by moving strings in 3d space)

These are not the same picture. One is  the other is

$\beta_{x,y}$ $\quad$ $\beta^{-1}_{y,x}$

They are not the same because there is no reason to.
Reversing the EH steps shows that they are invertible

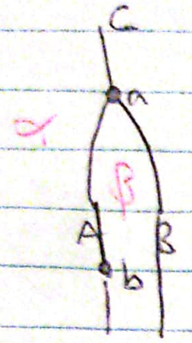


Topological remark:  in 3d. But w/ an extra dimension one could pull the crossing point for a 2nd and move the braiding over .

Similarly: had we one more direction for composition we could braid over and find $\beta_{xy} = \beta_{yx}^{-1}$.
This is the picture proof, but now it's easier to see the equational proof: (morally: use the new operation to insert an identity pushing the braid into it, swap it there.)

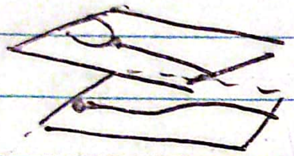
$$
\begin{tikzcd}
 & (1 \boxplus a) \otimes (b \boxplus 1) \arrow[r] & b \boxplus a \arrow[r] \arrow[d] & (b \otimes 1) \boxplus (1 \otimes a) \arrow[dr] & \\
a \otimes b \arrow[ur] \arrow[dr] & \text{by compatibility} & (1 \star a) \boxplus (b \star 1) \arrow[d] & \text{similarly} & b \otimes a \\
 & (a \boxplus 1) \otimes (1 \boxplus b) \arrow[r] & a \boxplus b \arrow[r] & (1 \otimes a) \boxplus (b \otimes 1) \arrow[ur] &
\end{tikzcd}
$$

We could keep going. A 2-category is a monoidal cat where we allow the codimension 0 strata to have labels



$\in \mathcal{B}$

A monoidal 2-category allows us to stack these sheets



~~then we can ad~~

+1 monoidal structure $\Rightarrow$ braiding
+2 monoidal structures $\Rightarrow$ syllepsis (red in the previous page becomes $\simeq$)
+3 monoidal structures $\rightarrow$ symmetry

application Given a ~~braided~~ braided monoidal n-category, ~~seen as a~~

- Say something about the diagram in page 5 defining the syllepsis

- table of k-tuply monoidal n-cats