Def $f:\mathbb{R}^n \to \mathbb{R}$ smooth, $x \in \mathbb{R}^n$

$$Df_x = (\partial_1 f, \dots, \partial_n f) \in \mathfrak{X}(\mathbb{R}^n)$$

$$D^2 f_x = \begin{pmatrix} \partial_1^2 f & \partial_1\partial_2 f & \cdots & \partial_1 \partial_n f \\ & \ddots & & \partial_2\partial_n f \\ & & \ddots & \vdots \\ & & & \partial_n^2 f \end{pmatrix} : \mathbb{R}^n \otimes \mathbb{R}^n \to \mathbb{R}$$

Def Let $x \in \mathbb{R}^n$ be a critical pt, i.e. $Df(x) = 0$

- $\mathrm{rank}(x) = \mathrm{rank}(D^2 f(x))$
- $\mathrm{index}(x) = \#$ negative eigenvalues of $D^2 f(x)$

x is nondegenerate if it has full rank and degenerate if not.

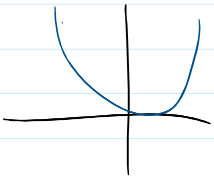
Thm **Morse Lemma** $x \in \mathbb{R}^n$ nondeg critical pt of $f:\mathbb{R}^n \to \mathbb{R}$ (WLOG $f(0)=0$, $x=0$)

Then $\exists\, \phi: \underset{x}{\underset{\cup}{U}} \to \mathbb{R}^n$ st $f\circ\phi: U \to \mathbb{R}^n$ is of the form

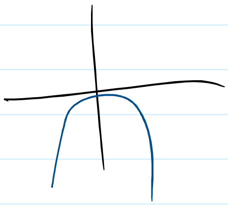
$$f\phi(u) = x_1^2 + \dots + x_{\mathrm{index}}^2 - x_{\mathrm{index}-1}^2 - \dots - x_n^2$$

Hence locally a critical point of $f:\mathbb{R}^n \to \mathbb{R}$ is equivalent to one of those 2^n possibilities. $n \geq 1$

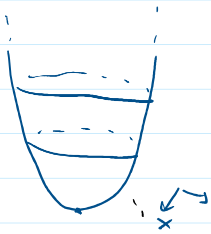
n=1 $f(x) = x^2$



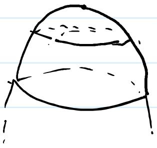
or $f(x) =$



n=2 $f(x) = x^2 + y^2$



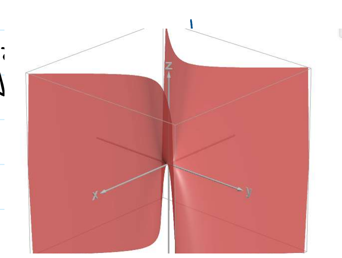
or $f(x) = -x^2 - y^2$



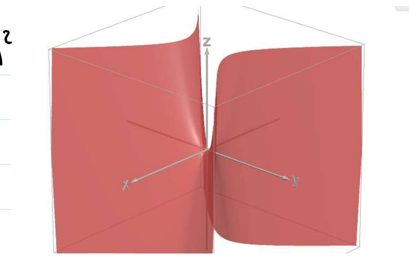
Note some level curves: $z = 10$ $\bigcirc$ $\qquad z = 10$ $\emptyset$

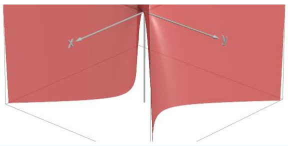
$z = -10$ $\emptyset$ $\qquad z = -10$ $\bigcirc$

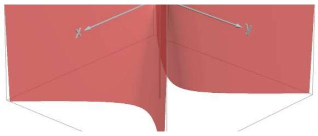
$f(x) = x^2 - y^2$



$f(x) = -x^2 + y^2$

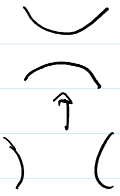






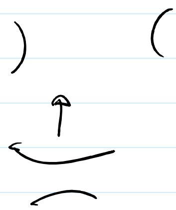
$z=10$

$z=-10$

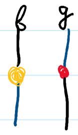


$z=10$

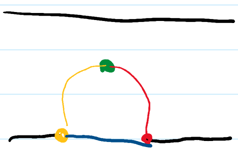
$z=-10$



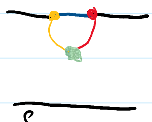
This is starting to look like an equivalence in a (∞,∞)-category.

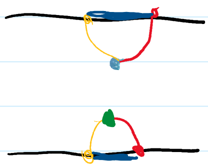


invertible: $fg \simeq 1_x$



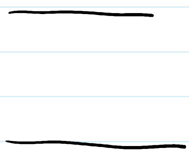
etc

But η is invertible too, i.e. $\exists$  ξ st $\xi\eta \simeq 1_{1_x}$ i.e.



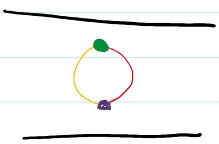
Θ

$\Rrightarrow$



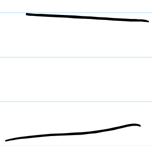
In homotopy.io:

Also $\eta\xi \simeq 1$

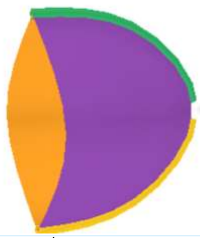


Δ

$\Rrightarrow$



In homotopy.io:



So there seems to be a correspondence between

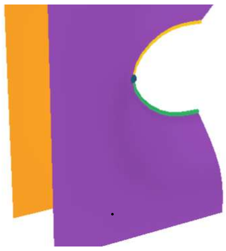
$\boxed{\text{normal forms of Morse functions in dim} \leq n} \longleftrightarrow \boxed{\text{n-morphisms appearing in an equivalence}}$ when coherent

Let's see if this continues. In the next step we would recall that Θ and Δ are invertible, let Θ^{-1} and Δ^{-1} be the inverses (I'm tired of making up letters)

In homotopy.io, Θ^{-1}/Δ^{-1} look the same as Θ/Δ but with reflection

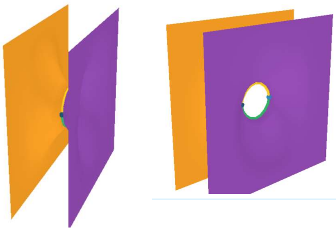
e.g. $\Theta^{-1} =$

Δ^{-1}



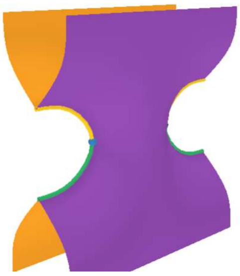

The compositions look like this:

$\Theta^{-1}\Theta$

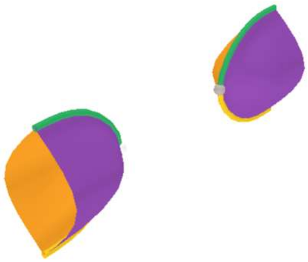


$\Delta\Delta^{-1}$

$\Theta\Theta^{-1}$



$\Delta^{-1}\Delta$

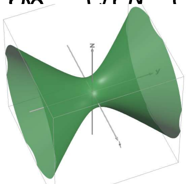


Now I will leave the ± 10 slices of 3d Morse functions

$$\pm x^2 \pm y^2 \pm z^2 = \pm 10$$

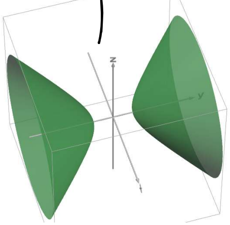
and move on for dramatic effect

$x^2-y^2+z^2=10$



$x^2+y^2+z^2=10$

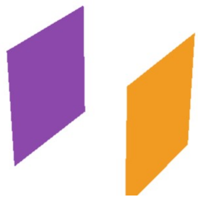
$x^2-y^2+z^2=-10$



$x^2+y^2+z^2=-10$

$\emptyset$

The other two will look similar but rotated. But if I draw a cap  in certain directions, I might have to interpret it as a sheet, like in the target of $\Theta\Theta^{-1} \Rightarrow 1_{\Theta\Theta}$



$\boxed{Q}$ Does this pattern extend?

$\boxed{Q}$ Does this pattern extend?

- Let $f: \mathbb{R}^{n+1} \to \mathbb{R}$ be a morse function, and $x \in \mathbb{R}$ a regular value. Can we interpret the n-manifold $f^{-1}(x) \in \mathbb{R}^{n+1}$ in an n-category?

- Let $f: \mathbb{R}^{n+1} \to \mathbb{R}$ be a morse function, and all $x \in [0,1]$ be regular values. Can we interpret $f^{-1}([0,1])$ as in a $(n+1)$-cat, as a morphism $f^{-1}(0) \to f^{-1}(1)$?

② DEGENERATE CRITICAL PTS

$\boxed{\text{Thm}}$ (Splitting lemma) Let $x \in \mathbb{R}^n$ be a degenerate critical pt of $f: \mathbb{R}^n \to \mathbb{R}$, with rank m and index p. Then locally: ⟹ in the same sense as the Morse lemma

$$f = \underbrace{x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_m^2}_{\substack{m\text{- variables} \\ \text{"non-degenerate" part} \\ \text{"Morse"}}} + \underbrace{g(x_{m+1}, \ldots, x_n)}_{\substack{n-m \text{ variables} \\ \text{"degenerate part"}}}, \text{ where } \operatorname{rank}(\nabla^2 g(x)) = 0$$

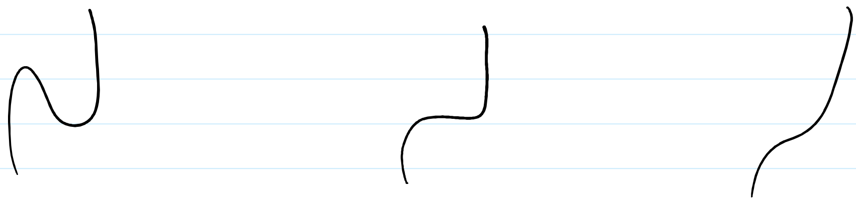
"fully degenerate"

⟹ to understand degeneracies, it suffices to understand functions with zero Hessian

e.g. $x^3: \mathbb{R} \to \mathbb{R}$ is degenerate at 0

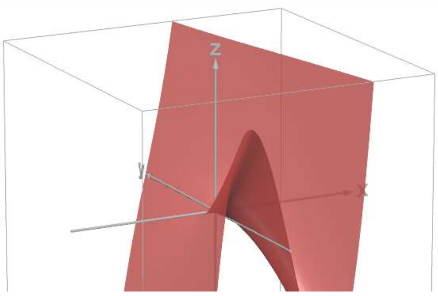
If we wiggle: $x^3 + ax^2 + bx + c$, a, b, c small, then we can cast this in $x^3 + ux$ form.

$u<0$ $\qquad$ $u=0$ $\qquad$ $u>0$



We can graph $z(x,y) = x^3 + yx$

This is clearly a triangle equation. Our goal is to understand what is going on.



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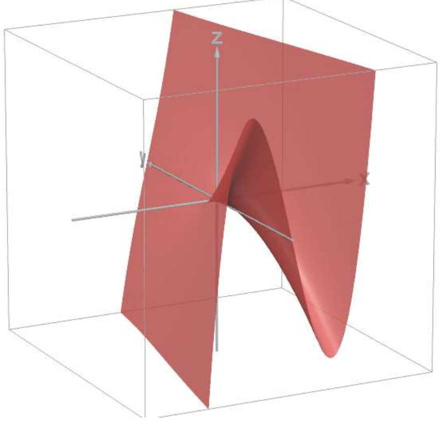
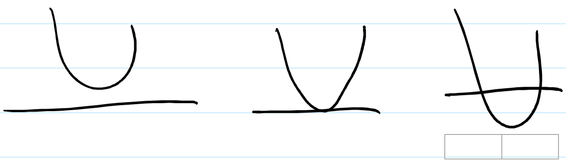


Another thing we can ask is the critical locus $\frac{d}{dx}(x^3+ux)=0 \Leftrightarrow 3x^2+u=0$

$u<0$	$u=0$	$u>0$
$\emptyset$	$-1/3$	$\sqrt{u/3}$

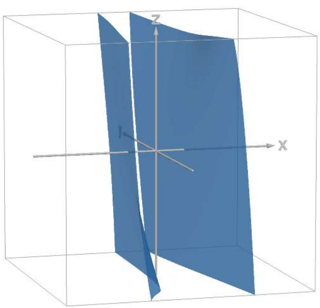
or if $u=y$ is a variable

$u<0$ $\quad$ $u=0$ $\quad$ $u>0$

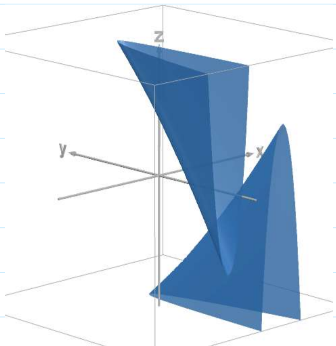


- Another one $x^4 : \mathbb{R} \to \mathbb{R}$. If we wiggle $x^4+ax^3+bx^2+cx+d$ ..

Anyway focus on x^4+ux^2+vx. We can't draw this in 3d, but we can get level surfaces



$w>0$



$w<0$

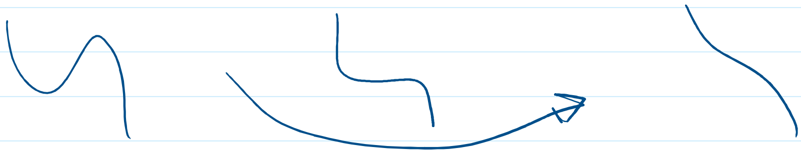
Not very enlightening. Again more interesting is the 0-locus of $\frac{d}{dx}(x^4+ux^2+vx)$.

In 3-d it's again the cusp $4x^3+2ux+v=0$

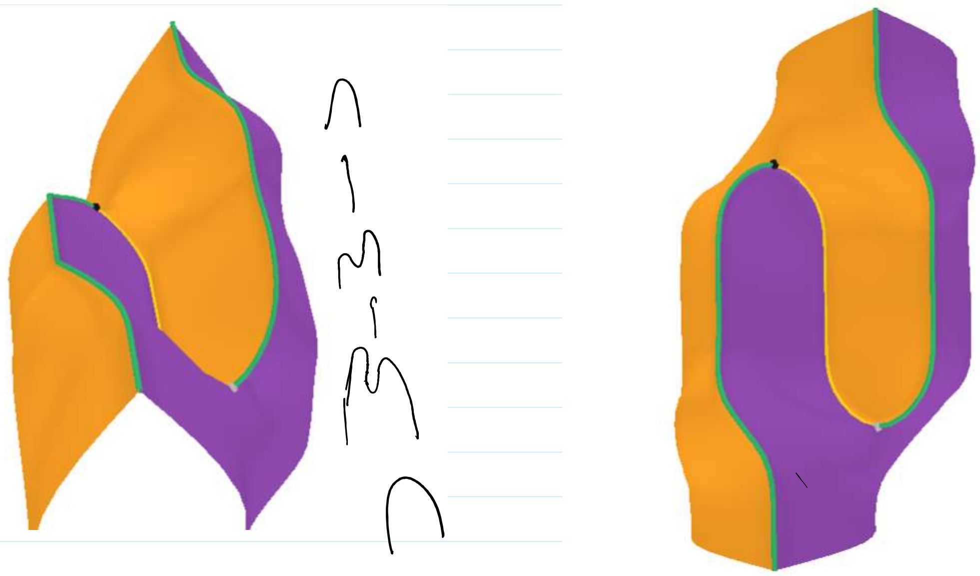
(Q) Which of these curves is the triangle equation?

Q Which of these cusps is the triangle equation?

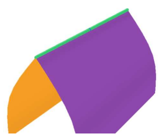
$u<0$ $u=0$ $u>0$



The swallowtail looks like this:



The equation is a 4-morphism from here to



Q How does $x^5 + ux^3 + vx^2 + wx$ gives rise to this stuff?