mod 2 cohomology of EM spaces (Serre)

Thm $H^\bullet(K(\mathbb{Z}/2,n);\mathbb{Z}/2) = \mathbb{Z}/2\,[Sq^{i_1}\cdots Sq^{i_r}(u)]$

where

$\to u$ is a generator of $H^n(K(\mathbb{Z}/2,n),\mathbb{Z}/2) = \mathbb{Z}/2$

$\to i_1 - i_2 - \cdots - i_r < n$

only $\pi_n = \mathbb{Z}/2$

Why care?

I COHOMOLOGY OPERATIONS

- Unstable: $H^n(X;G) \xrightarrow{\theta_X} H^{n+q}(X;H)$ natural in X

$$\mathrm{Nat}(H^n(-,G), H^{n+q}(-,H)) \overset{B}{\simeq} \mathrm{Nat}([-,K(G,n)], H^{n+q}(-,H))$$

coh. ops.

$$\overset{\text{Y}}{\simeq} H^{n+q}(K(G,n),H)$$

coh. classes

- Stable: $H^n(X;G) \xrightarrow{\theta_n} H^{n+q}(X,H)$ commuting w/suspension

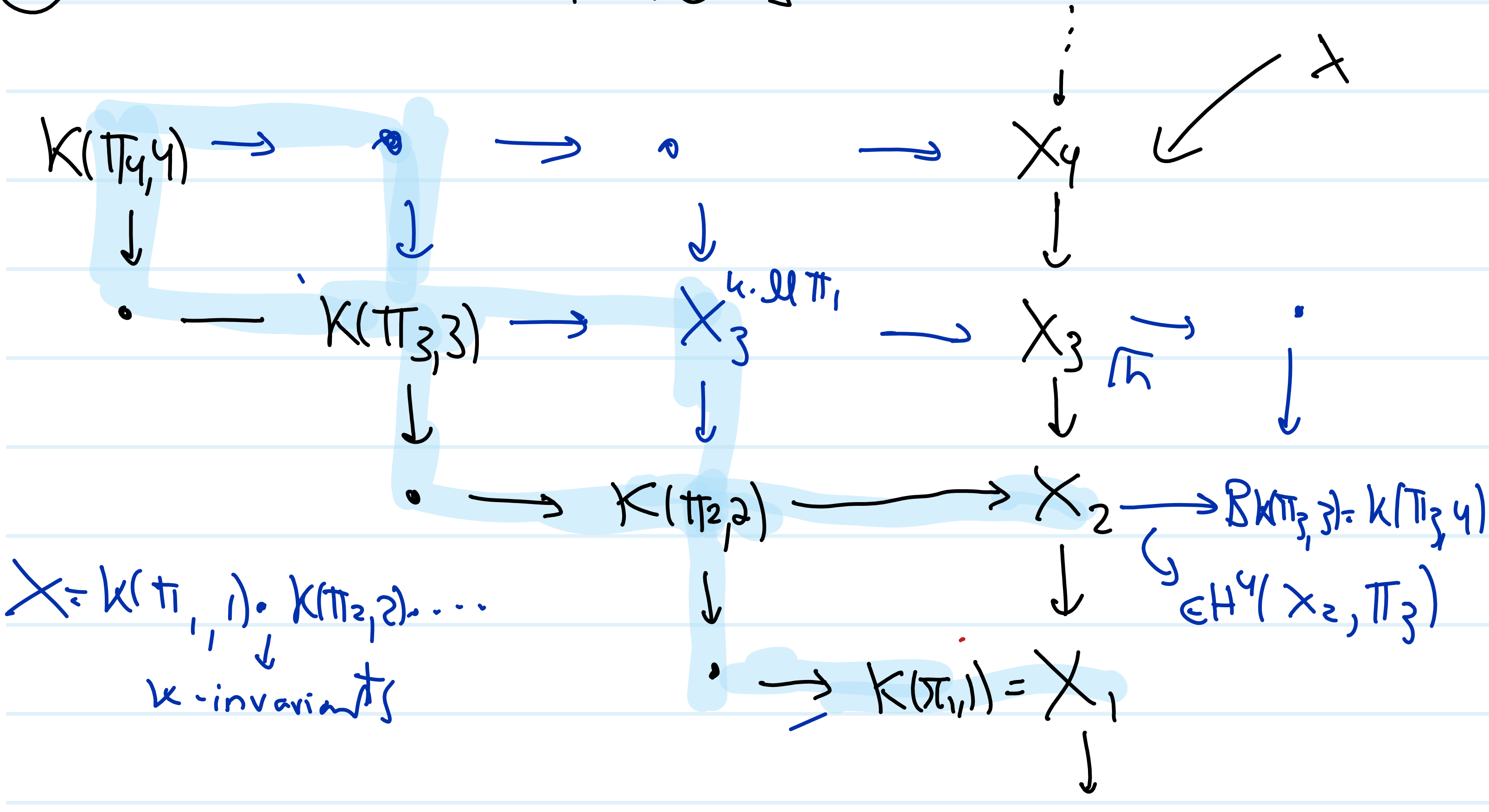
$$\wr$$

$$\varprojlim(\cdots \to H^{n+q+1}(K(G,n+1),H) \to H^{n+q}(K(G,n),H) \to \cdots)$$

$\wr$ Freudenthal

$$H^{N+q}(K(G,N),H),\quad N > q$$

II POSTNIKOV SYSTEMS



$K(\mathbb{Z}/p,1) = L^p$

p' is stable

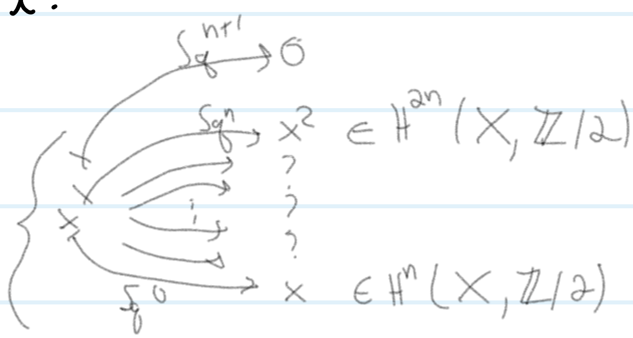
1 STEENROD SQUARES

for each $i \geq 0$

Def $Sq^i : H^n(X,\mathbb{Z}/2) \to H^{n+i}(X,\mathbb{Z}/2)$

↳ cohomology operations s.t.

① $Sq^i(x) = \begin{cases} 0, & \deg(x) < i \\ x^2, & \deg(x) = i \\ x, & 0 = i \end{cases}$



② Sq^i are stable ↳ commutes w/suspension

③ etc.

Def An *iterated Steenrod square* is a composite $\rightsquigarrow e(I)=1$

$$Sq^{i_1}\circ\cdots\circ Sq^{i_r} =: Sq^I;$$

This is *admissible* if $i_n \geq 2i_{n+1}$.

$Sq^4Sq^2Sq^1$ ✓

Sq^3Sq^2 ×

The *excess* of an *admissible* Sq^I is the number

$$e(I) = i_1 - i_2 - \cdots - i_r$$

$Sq^5Sq^2Sq^1 \to e(I)=2$

$= (i_1-2i_2)+(i_2-2i_3)+\cdots+(i_{r-1}-2i_r)+i_r \geq 0$

We can understand the statement of the theorem: $e(I)=0 \Leftrightarrow Sq^0$

Thm $H^*(K(\mathbb{Z}/2,n),\mathbb{Z}/2) = \mathbb{Z}/2[Sq^I(u)]$

where
- u is a generator of $H^n(K(\mathbb{Z}/2,n),\mathbb{Z}/2)$ $= \mathbb{Z}/2$ (Hurewicz)
- I ranges over admissible sequences with $e(I)<n$

$Sq^I(u) \in H^{i_1+\cdots+i_r+n}(K(\mathbb{Z}/2,n))$

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② TRANSGRESSION IN THE SERRE S.S.

A fibration $F \to X \to B$;

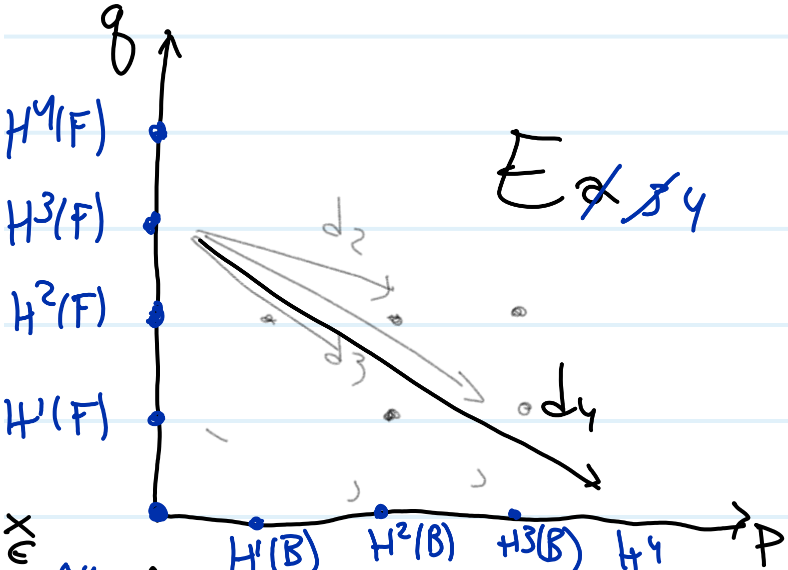
$$H^n(B) \longrightarrow H^n(F)$$

The *transgression* will be a partially defined map

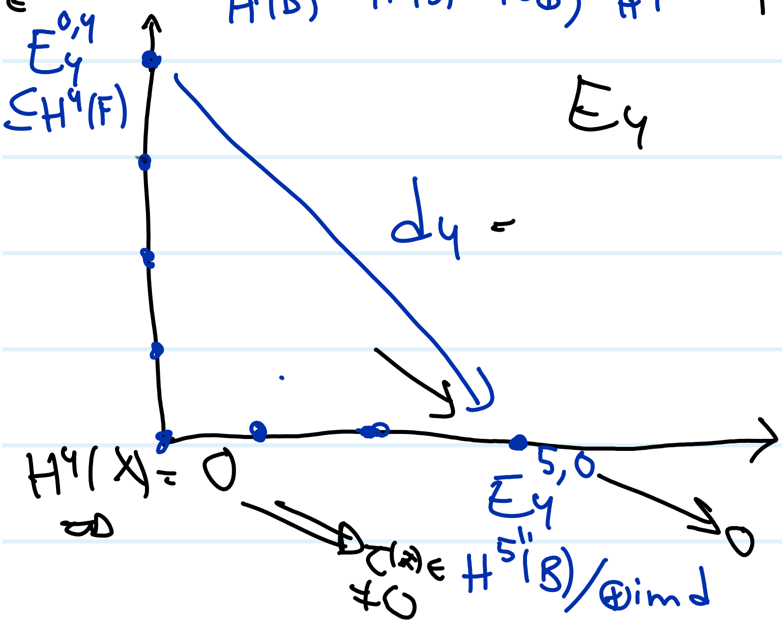
$$\tau \subseteq H^n(F) \longrightarrow H^{n+1}(B)/\sim$$

defined through the Serre spectral sequence of the fibration.

Suppose that $E_2^{p,q} = H^p(B) \otimes H^q(F)$



We turn the page by taking cohomology $E_{n+1}^{p,q} = \ker d / \operatorname{im} d$



As we turn the pages, we occasionally reach the "last" differential from the y- to the x-axis.

This is the transgression.

We want to understand the domain/codomain of τ:

- DOMAIN: there is no image in $\ker d/\operatorname{im} d$, so to turn the page is to take kernels.

  $\boxed{\text{Def}}$ $x \in E_2^{p,q}$ is transgressive if $d_3 x = \ldots = d_{n-1} x = 0$.

- CODOMAIN: everything is in the kernel in $\ker d/\operatorname{im} d$, so to turn the page is to take a quotient $H^n(B)/\operatorname{im} d$.

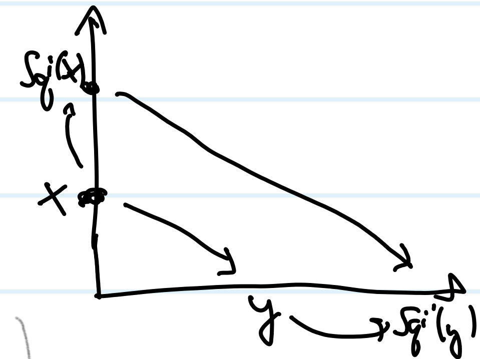
$\boxed{\text{Def}}$ The image of a transgressive $x \in H^n(F)$ is any $y \in \tau(x)$.

$\hookrightarrow \tau : \left\{ \begin{matrix} \text{transgressive} \\ \text{elements} \end{matrix} \right\} \longrightarrow H^{n+1}(B) / \oplus \operatorname{im} d.$
$\subseteq H^n(F)$

$\boxed{\text{Fact}}$ If x transgresses to y, then

- $Sq^i(x)$ is transgressive
- it transgresses to $Sq^i(y)$.

(this is true for any stable operation)



③ BOREL'S THM & $H^*(K(\mathbb{Z}/2, n), \mathbb{Z}/2)$

$\to$ e.g. $H^*(X, \mathbb{K})$

Def A *simple system of generators* for a graded algebra is a list $x_1, x_2, \cdots$ of generators s.t.

- each x_i is homogeneous
- the products $x_{i_1} * x_{i_2} * \cdots * x_{i_r}$, $i_1 < \cdots < i_r$ form a basis

$a_1 \wedge a_2$, $a_2 \wedge a_3 \wedge a_5$, etc.

e.g. $\Lambda[a_1, \cdots, a_n] \to \{a_1, \cdots, a_n\}$ is a simple system of generators

$\mathbb{K}[x] \longrightarrow \{x, x^2, x^4, \ldots\}$ "

$\mathbb{K}[y_1, \cdots, y_n] \to \{y_i, y_i^2, y_i^4, \ldots\}$ "

Thm (Borel) Let $F \to X \to B$ be a fibration s.t.

- $E_2^{p,q} = H^p(B) \otimes H^q(F)$ e.g. $\pi_1(B) = 0$
- $\tilde{H}^*(X;\mathbb{Z}/2) = 0$ $X = *$
- $H^*(F;\mathbb{Z}/2)$ has a simple system of transgressive generators (x_i) $\in H^n(F)$, $\in H^{n+1}(B)$

Suppose that the x_i transgress to y_i.

Then $H^*(B;\mathbb{Z}/2) = \mathbb{Z}/2[y_1, y_2, \dots]$.

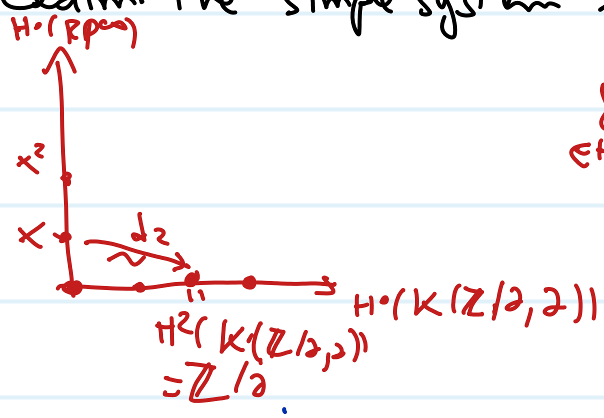
Cor $H^*(K(\mathbb{Z}/2,2)) = \mathbb{Z}/2[\underset{\in H^2}{u}, Sq^1(u), Sq^2 \cdot Sq^1(u), Sq^4 \cdot Sq^2 \cdot Sq^1(u), \dots]$

Pf) take the fibration $K(\mathbb{Z}/2,1) \to \text{path}\ * \longrightarrow K(\mathbb{Z}/2,2)$ ($F \longrightarrow X \longrightarrow B$)

$K(\mathbb{Z}/2,1) = \mathbb{RP}^\infty \rightsquigarrow H^*(\mathbb{RP}^\infty;\mathbb{Z}/2) = \mathbb{Z}/2[x]$

Claim: the simple system $x, x^2 = Sq^1 x, x^4 = Sq^2(x^2), \dots$ is transgressive.

$$x \mapsto y, \quad x^2 \mapsto Sq^1(y), \quad x^4 \mapsto Sq^2 Sq^1(y), \dots \qquad y \in H^2$$

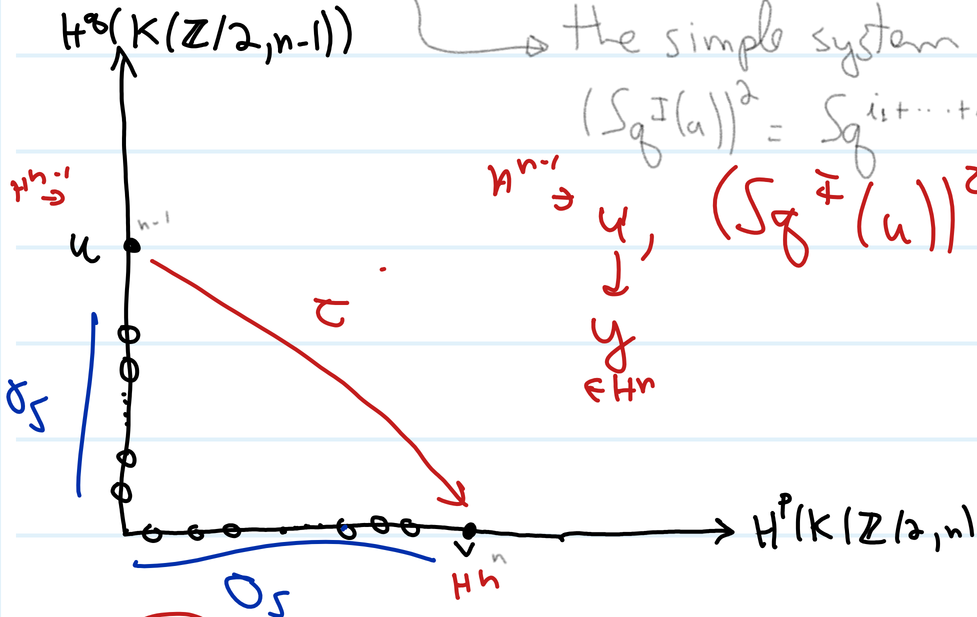


Borel $\to \mathbb{Z}/2[y, Sq^1(y), Sq^2 Sq^1(y), \dots]$

Cor $H^*(K(\mathbb{Z}/2,1)) = H^*(\mathbb{RP}^\infty;\mathbb{Z}/2) = \mathbb{Z}/2[x]$

In general, suppose $H^*(K(\mathbb{Z}/2,n-1);\mathbb{Z}/2) = \mathbb{Z}/2[Sq^I(u)]$, $e(I) < n-1$.

The generators are transgressive in $K(\mathbb{Z}/2,n-1) \to \cdot \to K(\mathbb{Z}/2,n)$:



the simple system is $(Sq^I(u))^{2^j}$, $e(I) < n-1$.

$(Sq^I(u))^2 = Sq^{i_1+\cdots+i_r+n}\, Sq^I(u) = Sq^{I'}(u)$, $e(I') < n$

$H^{n-1} \ni u$, $(Sq^I(u))^2 = Sq^{I'}(u)$, $e(I) < n-1$, $e(I') < n$

$u \mapsto y \in H^n$, $\quad Sq^{I'}(u) \mapsto Sq^{I'}(y)$

$$\Rightarrow \boxed{H^*(K(\mathbb{Z}/2,n),\mathbb{Z}/2) = \mathbb{Z}/2[(Sq^I(u))^{2^j}],\ e(I) < n-1,\ j \geq 0 \\ = \mathbb{Z}/2[Sq^J(u)],\ e(J) < n}$$

— " —

Similarly: $S^1 = K(\mathbb{Z},1) \to \cdot \to K(\mathbb{Z},2)$

$\boxed{\text{Thm}}$ $H^*(K(\mathbb{Z},n);\mathbb{Z}/2) = \mathbb{Z}/2[Sq^I(u)]$, $e(I) < n$ & Sq^I has no Sq^1

$\boxed{\text{Thm}}$ $H^*(K(\mathbb{Z}^{2^j},n);\mathbb{Z}/2) = \mathbb{Z}/2[Sq^I(u), Sq^J(v)]$ + conditions on I & J

$\boxed{\text{Cor}}$ $H^*(K(A,n);\mathbb{Z}/2)$ is determined (for f.g. A)

$K(\oplus \mathbb{Z}_{p_i} \oplus \mathbb{Z})$

④ HOMOTOPY GROUPS OF SPHERES

Whitehead

$$\begin{tikzcd} & \vdots \arrow[d] \\ K(\pi_{n+1}, n) \arrow[r] & (S^n, n+2) \arrow[d] \\ K(\pi_n, n-1) \arrow[r] & (S^n, n+1) \arrow[d] \\ & (S^n, n) = S^n \end{tikzcd} \qquad \begin{tikzcd} K(\pi_{n+2}, n+2) \arrow[r] & X_{n+2} \arrow[d] \\ K(\pi_{n+1}, n+1) \arrow[r] & X_{n+1} \arrow[d] \\ & K(\mathbb{Z}, n) \end{tikzcd}$$

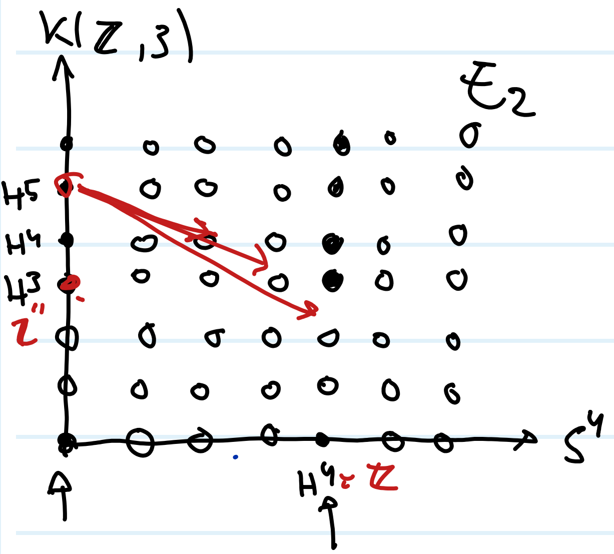
$\pi_{n+1}(S^n) = H^{n+1}(S^n, n+1)$ $\quad \hookrightarrow \pi_n = 0$

Prop $\pi_5(S^4) = \mathbb{Z}/2$

$\hookrightarrow H^5(S^4, 5) = \pi_5$

Pf : $K(\mathbb{Z}, 3) \longrightarrow (S^4, 5) \longrightarrow S^4$

$\pi_4 = 0$ $\quad \hookrightarrow$ no H^3, no H^4, $H^5 = \pi_5$



$H^3((K, 3), \mathbb{Z}) = \mathbb{Z}$

$H^4((K, 3), \mathbb{Z}) = 0$

$H^5(S^4, 5) = \pi_5(S^4)$

UCT: $H^5(K(\mathbb{Z}, 3), \mathbb{Z}/2) = \boxed{H^5(\mathbb{Z}, 3) \otimes \mathbb{Z}/2} \oplus \mathrm{Tor}(\underbrace{H^6(K(\mathbb{Z}, 3)}_{= \mathbb{Z}}, \mathbb{Z}/2)$

$\mathbb{Z}/2 \quad = \quad A \otimes \mathbb{Z}/2$

$Sq^2(u)$ $\hookrightarrow$ deg 3

$\hookrightarrow A = \cancel{\mathbb{Z}}$.

$A = \mathbb{Z}/2$

$A = \cancel{\mathbb{Z} \times \mathbb{Z}}/n$ n odd

$A = \cancel{\mathbb{Z}/2 \times \mathbb{Z}}/n$ n odd

$$\mathbb{C}P^\infty = K(\mathbb{Z},2) \to \cdot \to K(\mathbb{Z},3)$$