

LADDERS OF 2-CATEGORIES

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Warning: extremely work in progress.

Introduction

The cobordism hypothesis asserts that fully extended TQFTs with target in a symmetric (∞, n) -category C are classified by its fully dualizable objects.

To check dualizability it suffices [RV16] to study existence of adjoints in the bicategories $h_2^{(k)}(\mathcal{C})$ that have

- k-morphisms as objects.
- \bullet (k + 1)-morphisms as morphisms.
- classes of (k+2)-morphisms as 2-morphisms.

These form our prime example of a **ladder of bicat**egories. These are sequences of 2-dimensional categories

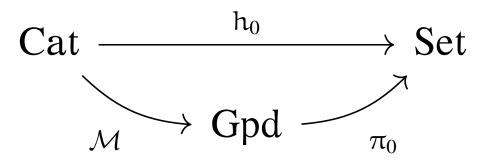
$$\mathcal{B}_0, \quad \mathcal{B}_1, \quad \ldots, \quad \mathcal{B}_k$$

that satisfy $h_1(\mathcal{B}_k) = Mor(\mathcal{B}_{k-1})$.

Preliminary results

Ladders of strict n-categories

Lemma. The functor h_0 (and h_1 , etc.) preserves filtered colimits:



Moreover Mor is an isofibration.

Corollary. The category Lad_n is accessible [B84].

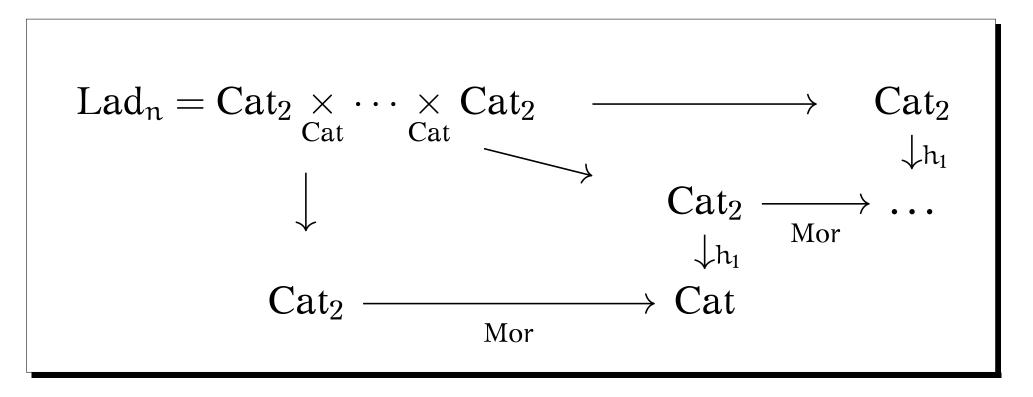
Ladders of gaunt categories

In this project we seek the extent of higher category theory which can be captured through ladders.

Methods

Ladders as pullbacks

We can model the category of ladders as a "pullback":



Homotopy 2-categories

The commutativity of the diagrams

$$\begin{array}{c|c} \operatorname{Cat}_{n+1} \xrightarrow{h_2^{(k+1)}} & \operatorname{Cat}_2 \\ & & & & \\ h_2^{(k)} & & & & \\ & & & & \\ \end{array} \cdot$$

An n-category is *gaunt* if its only invertibles are identities. In this context h_0 : Gaunt \rightarrow Set is right adjoint (to the discrete category functor).

Proposition. The category of Lad_n^g of ladders of gaunt 2-categories is locally presentable. The functor \mathbb{E}^g : $Gaunt_{n+1} \rightarrow Lad_n^g$ is a right adjoint.

Other directions

Ladders of 2-fold Segal spaces

Modelling $(\infty, 2)$ -categories through 2-fold Segal spaces we can define analogues of Mor and h_1 :

$$Mor: X_{\bullet\bullet} \mapsto X_{1\bullet} \qquad "h_1": X_{\bullet\bullet} \mapsto X_{\bullet 0} \qquad (2)$$

Due to the strong analogy between completeness and gauntness we expect a good theory of ladders in this case. In particular, we hope for a left adjoint

$$\mathbb{E}: \operatorname{Lad}_{n}^{\operatorname{CSS}} \to \operatorname{CSS}_{n+1}.$$

Example. The ladder of the (∞, n) -category Bord_{n,...,0} consists of the $(\infty, 2)$ -categories Bord_{d-2,d-1,d} (with

$$\operatorname{Cat}_2 \xrightarrow{}_{\operatorname{Mor}} \operatorname{Cat}$$

induces a functor $E: Cat_{n+1} \rightarrow Lad_n$.

Example. The ladder of a symmetric 2-category $(\mathcal{B}, \otimes, 1)$, seen as a 3-category with one object, consists of the monoidal category $h_1\mathcal{B}$ and the 2-category \mathcal{B} itself.

The functors involved stem from

- Mor : $Cat_{n+1} \rightarrow Cat_n$, in particular when n = 1.
- $\bullet \ h_0: Cat_n \to Set, \ and \ the functors \ h_k: Cat_{n+k} \to Cat_k$ it induces.

Proposition. A left adjoint to Mor is given by a "componentwise" suspension functor $L : Cat_n \rightarrow Cat_{n+1}$.

corners) for $2 \leq d \leq n$.

(1)

Symmetry and loops

The functor Mor that we defined is not really "categorical": it is not a well-defined functor on the (2,1)category of categories.

There are alternatives. One of them is to move to the context of symmetric n-categories, replacing Mor with the loops functor $\Omega: Cat_{n+1}^{\otimes} \to Cat_n^{\otimes}$.

References

- [B84] Bird, G. (1984). Limits in 2-categories of locally presentable categories [Ph.D. thesis]. University of Sydney.
- [DSPS13] Douglas, C.L., Schommer-Pries, C.J., & Snyder, N. (2013). Dualizable tensor categories. Memoirs of the American Mathematical Society.
- [RV16] Riehl, E. & Verity, D. (2016). Homotopy coherent adjunctions and the formal theory of monads. Advances in Mathematics 286 (2016) 802–888