Ladders of 2-categories

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Warning: extremely work in progress.

Introduction

The cobordism hypothesis asserts that fully extended TQFTs with target in a symmetric (∞, n)-category $\mathcal{C}$ are classified by its fully dualizable objects.

To check dualizability it suffices [RV16] to study existence of adjoints in the bicategories $h_2^{(k)}(\mathcal{C})$ that have

- k-morphisms as objects.
- $(k+1)$-morphisms as morphisms.
- classes of $(k+2)$-morphisms as 2-morphisms.

These form our prime example of a **ladder of bicategories**. These are sequences of 2-dimensional categories

$$\mathcal{B}_0, \quad \mathcal{B}_1, \quad \dots, \quad \mathcal{B}_k$$

that satisfy $h_1(\mathcal{B}_k) = \mathrm{Mor}\,(\mathcal{B}_{k-1})$.

In this project we seek the extent of higher category theory which can be captured through ladders.

Methods

Ladders as pullbacks

We can model the category of ladders as a "pullback":

$$\begin{array}{ccccc} \mathrm{Lad}_n = \mathrm{Cat}_2 \underset{\mathrm{Cat}}{\times} \cdots \underset{\mathrm{Cat}}{\times} \mathrm{Cat}_2 & & \longrightarrow & & \mathrm{Cat}_2 \\ & \searrow & & & \downarrow h_1 \\ \downarrow & & \mathrm{Cat}_2 & \xrightarrow{\mathrm{Mor}} & \dots \\ & & \downarrow h_1 & & \\ \mathrm{Cat}_2 & \xrightarrow{\mathrm{Mor}} & \mathrm{Cat} & & \end{array}$$

Homotopy 2-categories

The commutativity of the diagrams

$$\begin{array}{ccc} \mathrm{Cat}_{n+1} & \xrightarrow{h_2^{(k+1)}} & \mathrm{Cat}_2 \\ {\scriptstyle h_2^{(k)}}\downarrow & & \downarrow{\scriptstyle h_1} \\ \mathrm{Cat}_2 & \xrightarrow[\mathrm{Mor}]{} & \mathrm{Cat} \end{array} \quad . \tag{1}$$

induces a functor $Ł : \mathrm{Cat}_{n+1} \to \mathrm{Lad}_n$.

Example. The ladder of a symmetric 2-category $(\mathcal{B}, \otimes, \mathbf{1})$, seen as a 3-category with one object, consists of the monoidal category $h_1\mathcal{B}$ and the 2-category $\mathcal{B}$ itself.

The functors involved stem from

- $\mathrm{Mor} : \mathrm{Cat}_{n+1} \to \mathrm{Cat}_n$, in particular when $n = 1$.
- $h_0 : \mathrm{Cat}_n \to \mathrm{Set}$, and the functors $h_k : \mathrm{Cat}_{n+k} \to \mathrm{Cat}_k$ it induces.

Proposition. A left adjoint to Mor is given by a "componentwise" suspension functor $L : \mathrm{Cat}_n \to \mathrm{Cat}_{n+1}$.

Preliminary results

Ladders of strict n-categories

Lemma. The functor h_0 (and h_1, etc.) preserves filtered colimits:

$$\begin{array}{ccccc} \mathrm{Cat} & & \xrightarrow{h_0} & & \mathrm{Set} \\ & \searrow_{\mathcal{M}} & & \nearrow_{\pi_0} & \\ & & \mathrm{Gpd} & & \end{array}$$

Moreover Mor is an isofibration.

Corollary. The category Lad_n is accessible [B84].

Ladders of gaunt categories

An n-category is *gaunt* if its only invertibles are identities. In this context $h_0 : \mathrm{Gaunt} \to \mathrm{Set}$ is right adjoint (to the discrete category functor).

Proposition. The category of Lad_n^g of ladders of gaunt 2-categories is locally presentable. The functor $Ł^g : \mathrm{Gaunt}_{n+1} \to \mathrm{Lad}_n^g$ is a right adjoint.

Other directions

Ladders of 2-fold Segal spaces

Modelling $(\infty, 2)$-categories through 2-fold Segal spaces we can define analogues of Mor and h_1:

$$\mathrm{Mor} : X_{\bullet\bullet} \mapsto X_{1\bullet} \qquad \text{"}h_1\text{"} : X_{\bullet\bullet} \mapsto X_{\bullet 0} \tag{2}$$

Due to the strong analogy between completeness and gauntness we expect a good theory of ladders in this case. In particular, we hope for a left adjoint

$$Ł : \mathrm{Lad}_n^{\mathrm{CSS}} \to \mathrm{CSS}_{n+1}.$$

Example. The ladder of the (∞, n)-category $\mathrm{Bord}_{n,\dots,0}$ consists of the $(\infty, 2)$-categories $\mathrm{Bord}_{d-2,d-1,d}$ (with corners) for $2 \le d \le n$.

Symmetry and loops

The functor Mor that we defined is not really "categorical": it is not a well-defined functor on the (2,1)-category of categories.

There are alternatives. One of them is to move to the context of symmetric n-categories, replacing Mor with the loops functor $\Omega : \mathrm{Cat}_{n+1}^{\otimes} \to \mathrm{Cat}_n^{\otimes}$.

References

[B84] Bird, G. (1984). Limits in 2-categories of locally presentable categories [Ph.D. thesis]. University of Sydney.

[DSPS13] Douglas, C.L., Schommer-Pries, C.J., & Snyder, N. (2013). Dualizable tensor categories. Memoirs of the American Mathematical Society.

[RV16] Riehl, E. & Verity, D. (2016). Homotopy coherent adjunctions and the formal theory of monads. Advances in Mathematics 286 (2016) 802–888

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