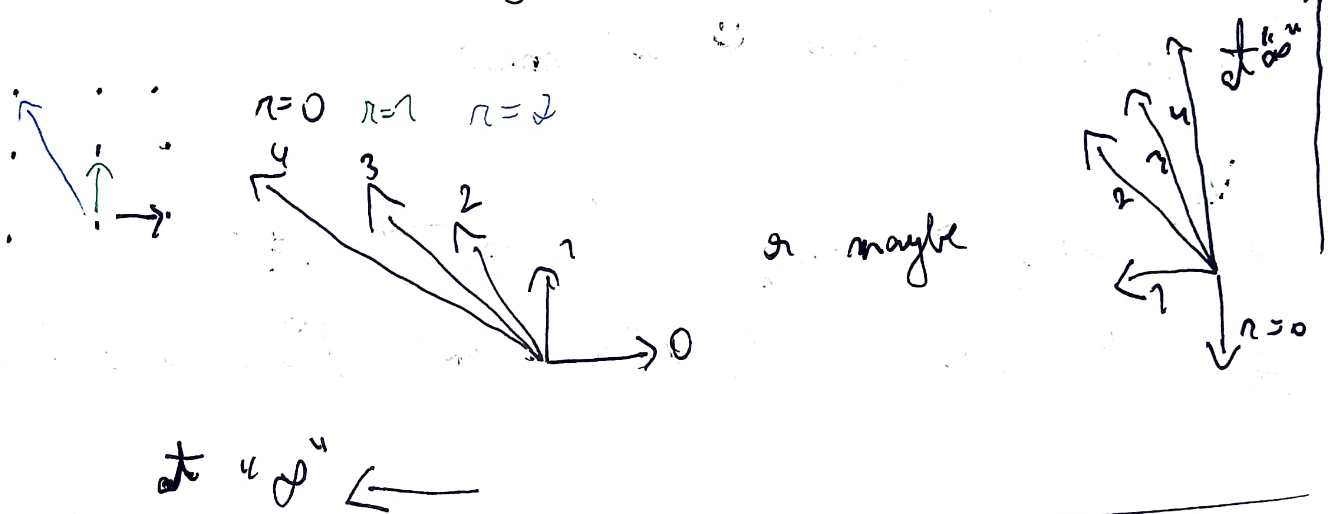
Abstract definition: A Spectral Sequence is a family of objects $\{E^r_{p,q}\}$ of $\mathcal{C}$ for all $p,q \in \mathbb{Z}$, $\underline{r \geq a}$

↳ Last week $a=2$, Today $a=0$

w/ maps $d^r_{p,q} : E^r_{pq} \to E^r_{p-r, q+r-1}$ s.t $d^r \circ d^r = 0$ and

$$\underbrace{E^{r+1}_{pq} \cong H(E^r_{pq}) = \frac{\ker d^r_{pq}}{\operatorname{im}(d^r_{p+r, q-r+1})}}$$

Turning pages = Taking homology



1st quadrant SS: $E^r_{pq} = 0 \ \forall p,q < 0$ for r large enough the maps have either 0 as a target or source.

Can define 2nd, 3rd, 4th quadrant SS similarly.

1 ② Vague

Def: (Convergence). ~~At some point we~~ For every p,q, $\exists\, r$ s.t. $E^{r'}_{p,q} = E^{r''}_{p,q}$ $\forall\, r', r'' > r$. we denote by $E^{\infty}_{p,q}$

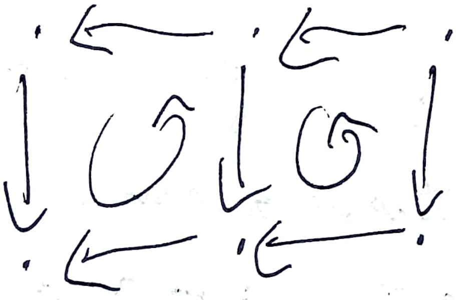
Convergence means a colimit of some of the $E^{\infty}_{p,q}$ for p,q satisfying some properties.

<u>Double complexes</u> (more specific, more concrete)

Def: A double complex $C_{\cdot\cdot}$ is a family of objects $(C_{pq})_{p,q\in\mathbb{Z}}$ of ${}_R\mathrm{Mod}$ (☺) w/ maps

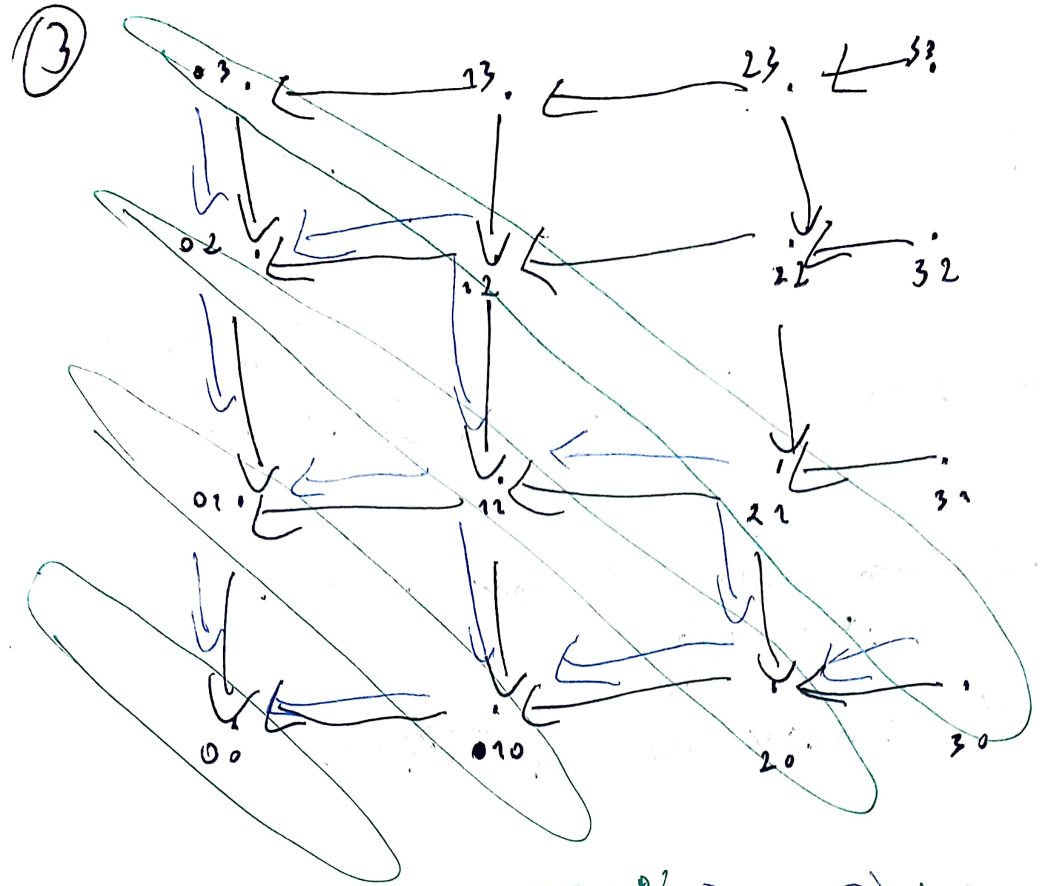
$$d^h : C_{p,q} \to C_{p-1,q}, \quad d^v : C_{pq} \to C_{pq-1}$$

s.t. $d^h \circ d^h = 0$, $d^v \circ d^v = 0$ and $d^v \circ d^h = \pm\, d^h \circ d^v$

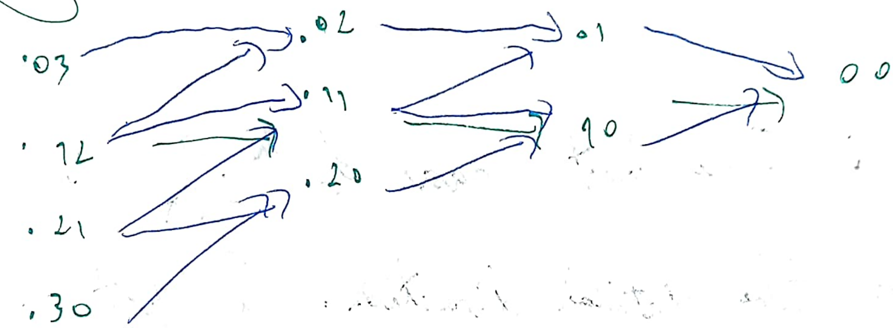


Def: The total complex ~~is~~ of an anticommutative double complex is $\mathrm{Tot}_n(C_{\cdot\cdot}) = \bigoplus_{p+q=n} C_{pq}$ w/ differential $d = d^v + d^h$

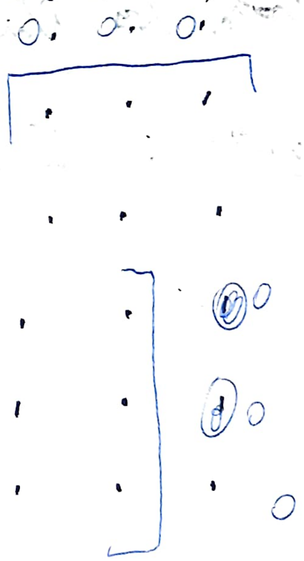
(3)



Tot



We can define 2 Filtrations on $\mathrm{Tot}(C_{\cdot\cdot})$



Rows: $F_p^{(R)}\mathrm{Tot}_m(C_{\cdot\cdot}) = \bigoplus_{i\le p} C_{i,m-i}$

column: $F_q^{(C)}\mathrm{Tot}_m(C_{\cdot\cdot}) = \bigoplus_{j\le q} C_{m-j,j}$

Both cases: $F_p\mathrm{Tot}_\cdot(C_{\cdot\cdot}) / F_{p-1}\mathrm{Tot}_\cdot(C_{\cdot\cdot}) = C_{p\hat{q}}$

(4)

The idea: (i)

Step 1: Define your double complex $(E.G. \begin{cases} Hom(C., D.) \\ C. \otimes D. \end{cases})$

Step 2: Each filtration (rows, cols) give us a SS $^{h}E^{r}_{pq}$, $^{v}E^{r}_{pq}$

Step 3: They both converge to $H_{\cdot}(Tot_{\cdot}(C_{\cdot\cdot}))$

Prop 3.3: The first page of $^{h}E^{r}_{pq}$ is given by the homologies in the vertical direction. The first page of $^{v}E^{r}_{pq}$ is given by the homologies in the horizontal direction.

Example: 5-lemma

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[r, "g"] \arrow[d, "\delta"'] & E \arrow[d, "\varepsilon"'] \\ A' \arrow[r, "f'"'] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r, "g'"'] & E' \end{tikzcd}$$

Exact rows

⑤ Flip to get SS on 1st quadrant

E^0

$$\begin{tikzcd}
0 & \operatorname{coker} g \arrow[l] \arrow[d] & E \arrow[l] \arrow[d, "\varepsilon"] & D \arrow[l, "g"'] \arrow[d, "\delta"] & C \arrow[l] \arrow[d, "\gamma"] & B \arrow[l] \arrow[d, "\beta"] & A \arrow[l, "f"'] \arrow[d, "\alpha"] & \operatorname{Ker} f \arrow[l] \arrow[d, "\alpha|_{\operatorname{Ker} f}"] & 0 \arrow[l] \\
0 & \operatorname{coker} g' \arrow[l] & E' \arrow[l] & D' \arrow[l, "g'"] & C' \arrow[l] & B' \arrow[l] & A' \arrow[l, "f'"] & \operatorname{Ker} f' \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

${}^{h}E^1$

$$\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix}$$

$\Rightarrow {}^{h}E^1_{pq} = {}^{h}E^{\infty}_{pq} = 0 \ \forall p,q$

$\Rightarrow {}^{v}E^{\infty}_{pq} = 0 \ \forall p,q$

${}^{v}E^1$

$$\begin{tikzcd}
0 & * \arrow[l] & \operatorname{Ker} \varepsilon \arrow[l] & \operatorname{Ker} \delta \arrow[l] & \operatorname{Ker} \gamma \arrow[l] & \operatorname{Ker} \beta \arrow[l] & \operatorname{Ker} \alpha \arrow[l] & * \arrow[l] & 0 \arrow[l] \\
0 & * \arrow[l] & \operatorname{coker} \varepsilon \arrow[l] \arrow[ull] & \operatorname{coker} \delta \arrow[l] \arrow[ull] & \operatorname{coker} \gamma \arrow[l] \arrow[ull] & \operatorname{coker} \beta \arrow[l] \arrow[ull] & \operatorname{coker} \alpha \arrow[l] \arrow[ull] & * \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

Assume β, δ are mono and α is an epi ($\operatorname{Ker} \beta = \operatorname{Ker} \delta = 0$ $= \operatorname{coker}$

${}^{v}E^1$

$$0 \leftarrow * \leftarrow \operatorname{Ker} \varepsilon \leftarrow 0 \leftarrow \operatorname{Ker} \gamma \leftarrow 0 \leftarrow \operatorname{Ker} \alpha \leftarrow * \leftarrow 0$$

$$0 \leftarrow * \leftarrow \operatorname{coker} \varepsilon \leftarrow \operatorname{coker} \delta \leftarrow \operatorname{coker} \gamma \leftarrow \operatorname{coker} \beta \leftarrow 0 \leftarrow * \leftarrow 0$$

⑥ Taking Homology to get $\tilde{E}^2$ we have:

$\tilde{E}^2$

$$\begin{array}{cccccccc}
* & * & * & \operatorname{Ker}\gamma & * & * & * & * \\
* & * & * & * & * & 0 & * & *
\end{array}$$

Note that after this page, every map is between 0s so everything has to die right now, in particular,

$\boxed{\operatorname{Ker}\gamma = 0}$ as desired

Now assume β, δ are epimorphisms and ε a monomorphism ($\operatorname{coker}\beta = \operatorname{coker}\delta = \operatorname{Ker}\varepsilon = 0$)

$$0 \leftarrow * \leftarrow 0 \leftarrow \operatorname{Ker}\delta \leftarrow \operatorname{Ker}\gamma \leftarrow \operatorname{Ker}\beta \leftarrow \operatorname{Ker}\alpha \leftarrow * \leftarrow 0$$

$$0 \leftarrow * \leftarrow \operatorname{coker}\varepsilon \leftarrow 0 \leftarrow \operatorname{coker}\gamma \leftarrow 0 \leftarrow \operatorname{Ker}\alpha \leftarrow * \leftarrow 0$$

Taking homology:

$\tilde{E}^2$

$$\begin{array}{ccccccccc}
* & * & 0 & * & * & * & * & * & * \\
* & * & * & * & \operatorname{coker}\gamma & * & * & * & *
\end{array}$$

$\Longrightarrow$ $\boxed{\operatorname{coker}\gamma = 0}$

Example 2: <u>Connecting map</u>

We want to prove the connecting map part of the snake lemma:

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' & \end{tikzcd}$$

Flipping it to get a 1st quadrant SS:

E^0

$$\begin{tikzcd} 0 & C \arrow[l] \arrow[d, "\gamma"] & B \arrow[l, "g"'] \arrow[d, "\beta"] & A \arrow[l, "f"'] \arrow[d, "\alpha"] & \ker f \arrow[l] & 0 \arrow[l] \\ 0 & \operatorname{coker} g' \arrow[l] & C' \arrow[l] & B' \arrow[l, "g'"] & A' \arrow[l, "f'"] & 0 \arrow[l] \end{tikzcd}$$

Rows are exact so again the SS converges to 0

${}^vE^1$

$$0 \leftarrow \ker \gamma \xleftarrow{\varphi} \ker \beta \leftarrow \ker \alpha \leftarrow \ker f \leftarrow 0$$

$$0 \leftarrow \operatorname{coker} g' \leftarrow \operatorname{coker} \gamma \leftarrow \operatorname{coker} \beta \xleftarrow{\psi} \operatorname{coker} \alpha \leftarrow 0$$

Diagram chasing shows every other spot is exact

${}^vE^2$

$$\begin{array}{ccccccc} 0 & \operatorname{coker} \psi & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \ker \varphi & 0 & 0 \end{array}$$

$\Rightarrow$ $\ker \varphi \cong \operatorname{coker} \psi$

(8) Putting everything together we have

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} f \arrow[r] & \operatorname{Ker} \alpha \arrow[r] & \operatorname{Ker} \beta \arrow[r, "\varphi"] & \operatorname{Ker} \gamma \arrow[r, dashed] \arrow[d, two heads] & \operatorname{coker} \alpha \arrow[r, "\psi"] & \operatorname{coker} \beta \\
 & & & & \operatorname{coker} \varphi \arrow[r, phantom, "\cong"] & \operatorname{Ker} \psi \arrow[u, hook] &
\end{tikzcd}$$

(i) from E^1

from E^2