1 If the individual does not buy insurance, then they will have wealth \$4,000 with probability .95, and \$2,000 with probability 0.05. The expected utility is therefore

$$0.95u(4000) + 0.05u(2000) = 0.95 \times \left(7000 \times 4000 - 4000^2\right) + 0.05 \times (7000 \times 2000 - 2000^2) = \$11,900,000$$

If the individual buys insurance, their wealth is guaranteed to be \$3,880, so their utility is $7000 \times 3880 - 3880^2 = \$12,105,600$. Therefore, the individual is better off buying the insurance.

Advantages of tort system	Advantages of no-fault system
Increases coverage costs for at-fault drivers, thus increasing the incentive to drive carefully.	Reduces litigation costs
More flexibility to taylor payments to injured parties needs. Under no-fault system, benefits usually defined by a formula.	Evidence shows that under tort system, small claims are overcompensated, whereas larger claims are undercompensated.

3 80% of the house price is $0.8 \times 350000 = \$280,000$, so the coinsurance pays $\frac{260000}{280000}$ of the loss, which is $\frac{260000 \times 70000}{280000} = \$65,000$.

4

a) This is less than the deductible, so no payment is made.

b) $2500 - 1000 = \$1,500$. The insurer therefore pays $0.8 \times 1500 = \$1,200$.

c) $101600 - 1000 = \$100,600$. The insurer therefore pays $0.8 \times 100600 = \$80,480$.

d) $146900 - 1000 = \$145,900$. $0.8 \times 145900 = \$116,720$. This is more than the policy limit, so the insurer pays the policy limit \$100,000.

5 (a) For a Pareto distribution, the probability that a loss exceeds x is $\left(1+\frac{x}{\theta}\right)^{-\alpha}$, so in this case, for any particular loss, it is 101^{-2}. Each policy has a 0.5 probability of producing a loss, and therefore a $\frac{1}{2\times 101^2}$ probability of producing a loss exceeding \$1,000,000. This means that the probability of a single loss exceeding \$1,000,000 is $1-\left(1-\frac{1}{2\times 101^2}\right)^{100}=0.004889607$.

(b) Calculating the exact probability is difficult.

For the Pareto distribution, the expectation of the limited loss random variable is

$$\begin{aligned}
\mathbb{E}(X\wedge 100000) &= \int_0^{100000} \frac{1}{\left(1+\frac{x}{10000}\right)^2}\,dx \\
&= 10000\int_0^{10} \frac{1}{(1+u)^2}\,du \\
&= 10000\int_1^{11} a^{-2}\,da \\
&= 10000\left[-a^{-1}\right]_1^{11} \\
&= 10000\left(1-\frac{1}{11}\right) \\
&= \frac{100000}{11}
\end{aligned}$$

We also calculate

$$\begin{aligned}
\mathbb{E}((X\wedge 100000)^2) &= \int_0^{100000} \frac{2x}{\left(1+\frac{x}{10000}\right)^2}\,dx \\
&= 10000^2\int_0^{10} \frac{2u}{(1+u)^2}\,du \\
&= 10000^2\int_1^{11} 2(a-1)a^{-2}\,da \\
&= 10000^2\int_1^{11} 2a^{-1}-2a^{-2}\,da \\
&= 2\times 10^8\left[\log(a)+a^{-1}\right]_1^{11} \\
&= 2\times 10^8\left(\log(11)-\frac{10}{11}\right)
\end{aligned}$$

This gives us

$$\begin{aligned}
\mathrm{Var}(X\wedge 100000) &= 10^8\left(2\log(11)-\frac{20}{11}-\frac{100}{121}\right) \\
&= 21,5116,245
\end{aligned}$$

Now the losses per policy are either X or 0 with probability 0.5. The expected loss per policy is therefore $\frac{50000}{11}$, while the variance of loss per policy is $\left(\frac{50000}{11}\right)^2 \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times 215116245 = 112,723,412$ The expected aggregate loss is therefore $\frac{5000000}{11} = \$454,545$ and the variance of aggregate loss is 11,272,341,200. The standard deviation is therefore \$106,171. The loss of \$1,000,000 is therefore 5.137496 standard deviations above the mean. If we use a normal approximation, the probability of aggregate losses exceeding \$1,000,000 is 1.392117×10^{-7}.

It is unclear how good the normal approximation is, so I also simulated 1,000,000 random aggregate losses, and found that in 82 of them, the aggregate loss exceeded \$1,000,000. Clearly, the normal approximation underestimates this probability, but it is still far less than the probability in (a).

6

If the attachment point is a, then the expected aggregate loss is $\theta(1-e^{-\frac{a}{\theta}})$, and the expected claim on the stop-loss insurance is $\theta e^{-\frac{a}{\theta}}$. The variance of the aggregate loss payment is

$$\begin{aligned}
\theta^2\left(\int_0^{\frac{a}{\theta}} x^2e^{-x}\,dx+\frac{a^2}{\theta^2}e^{-\frac{a}{\theta}}-(1-e^{-\frac{a}{\theta}})^2\right) &= \theta^2\left(\left[-x^2e^{-x}\right]_0^{\frac{a}{\theta}}+2\int_0^{\frac{a}{\theta}} xe^{-x}\,dx+\frac{a^2}{\theta^2}e^{-\frac{a}{\theta}}-1+2e^{-\frac{a}{\theta}}-e^{-2\frac{a}{\theta}}\right)\\
&= \theta^2\left(-\frac{a^2}{\theta^2}e^{-\frac{a}{\theta}}+2\left[-xe^{-x}\right]_0^{\frac{a}{\theta}}+2\int_0^{\frac{a}{\theta}} e^{-x}\,dx+\frac{a^2}{\theta^2}e^{-\frac{a}{\theta}}-1+2e^{-\frac{a}{\theta}}-e^{-2\frac{a}{\theta}}\right)\\
&= \theta^2\left(-\frac{a^2}{\theta^2}e^{-\frac{a}{\theta}}-2\frac{a}{\theta}e^{-\frac{a}{\theta}}+2\left(1-e^{-\frac{a}{\theta}}\right)+\frac{a^2}{\theta^2}e^{-\frac{a}{\theta}}-1+2e^{-\frac{a}{\theta}}-e^{-2\frac{a}{\theta}}\right)\\
&= \theta^2\left(1-2\frac{a}{\theta}e^{-\frac{a}{\theta}}-e^{-2\frac{a}{\theta}}\right)
\end{aligned}$$

The insurer's premium is therefore set at

$$P=\theta\left(2e^{-\frac{a}{\theta}}+1-e^{-\frac{a}{\theta}}+\sqrt{1-2\frac{a}{\theta}e^{-\frac{a}{\theta}}-e^{-2\frac{a}{\theta}}}\right)$$

We want to minimise this P. We substitute $u=\frac{a}{\theta}$ and calculate

$$\frac{dP}{du}=\theta\left(-e^{-u}+\frac{ue^{-u}-e^{-u}+e^{-2u}}{\sqrt{1-2ue^{-u}-e^{-2u}}}\right)$$

We find the minimum by setting this equal to zero:

$$\begin{aligned}
-e^{-u}+\frac{ue^{-u}-e^{-u}+e^{-2u}}{\sqrt{1-2ue^{-u}-e^{-2u}}} &= 0\\
\frac{ue^{-u}-e^{-u}+e^{-2u}}{\sqrt{1-2ue^{-u}-e^{-2u}}} &= e^{-u}\\
\frac{(ue^{-u}-e^{-u}+e^{-2u})^2}{1-2ue^{-u}-e^{-2u}} &= e^{-2u}\\
(u-1+e^{-u})^2 &= 1-2ue^{-u}-e^{-2u}\\
u^2-2u+4ue^{-u}-2e^{-u}+2e^{-2u} &= 0
\end{aligned}$$

Numerically, we solve this to get $u=0.9640863$, so the attachment point that minimises the premium is 0.9640863 times the mean aggregate claims.

7

(a) The premium period is from the 1st of October 2015 to the 30th of September 2016, so there are three months out of 12 in 2015. The earned premium is therefore $640 \times \frac{3}{12} = \$160$.

[If we divide by number of days, the earned premium is $640 \times \frac{91}{365} = \$159.56$.]

(b) There are nine months out of 12 in 2016, so the earned premium is $640 \times \frac{9}{12} = \$480$.

[If we divide by number of days, the earned premium is $640 \times \frac{274}{365} = \$480.44$.]

8

The new rate will cover annual policies written in 2018. These will be effective for 1 year. Assuming the time that policies are written is uniformly distributed over the year, the number of policies in force at the new rate t years from the start of 2018 is given by

$$\begin{cases} Nt & \text{if } t < 1 \\ N(2-t) & \text{if } 1 \leqslant t < 2 \\ 0 & \text{otherwise} \end{cases}$$

Assume that the rate of claims is proportional to the number of policies in force (so claims per policy does not depend on time of year). Assume also that claims in accident year 2016 were uniformly distributed over the year. We integrate over the inflation factor for all claim times. That is, if expected claim amount at the start of 2016 is C, then we have $C\int_0^1 (1.03)^t\,dt = 26000$, and we want to calculate

$$C(1.03)^2\left(\int_0^1 t(1.03)^t\,dt + \int_1^2 (2-t)(1.03)^t\,dt\right)$$

We have that

$$\int_0^1 (1.03)^t\,dt = \left[\frac{1.03^t}{\log(1.03)}\right]_0^1 = \frac{0.03}{\log(1.03)} = 1.014926$$

$$\int_0^1 t(1.03)^t\,dt = \left[t\frac{1.03^t}{\log(1.03)}\right]_0^1 - \int_0^1 \frac{1.03^t}{\log(1.03)}\,dt = \frac{1.03}{\log(1.03)} - \frac{0.03}{\log(1.03)^2} = 0.509963$$

$$\int_1^2 (2-t)(1.03)^t\,dt = 1.03\int_0^1 (1-s)(1.03)^s\,dt = 1.03\left(\frac{0.03}{\log(1.03)} - \frac{1.03}{\log(1.03)} + \frac{0.03}{\log(1.03)^2}\right) = 0.520112$$

$$\int_0^1 t(1.03)^t\,dt + \int_1^2 (2-t)(1.03)^t\,dt = \frac{0.03^2}{\log(1.03)^2}$$

We therefore get $1.014926C = 26000$ and the expected claim amount per claim for policy year 2018 is

$$(1.03)^2(0.509963 + 0.520112)C = \frac{(1.03)^2(0.509963 + 0.520112)}{1.014926} \times 26000 = 1.076735 \times 26000 = \$27,995.11$$

so the pure premium is $27995.11 \times 0.003 = \$83.99$.

[Algebraically, we can write the expected claim amount as $1.03^2 \times \frac{0.03}{\log(1.03)} \times 26000$.]

[As a sanity check for the calculated inflation factor, we have that the average claim time in accident year 2016 is the middle of the year. Inflation is therefore approximately $(1.03)^{\frac{1}{2}} = 1.01488915651$. The average claim time in policy year 2018, is the end of 2018, so inflation is approximately 1.03.]

9 The insurance line started $\frac{2}{12}=\frac{1}{6}$ of the way through the year. Assuming policies are sold throughout the year, the number of policies in force at time t is proportional to $t-\frac{1}{6}$. If the losses at time t are proportional to the number of policies in force, then the density function of the time of a random loss is $f(t)=\frac{72}{25}\left(t-\frac{1}{6}\right)=2.88\left(t-\frac{1}{6}\right)$. The expected inflation from 1st March 2022 to the time of a random claim in Accident Year 2022 is therefore

$$\begin{aligned}\int_0^{\frac{5}{6}} 2.88t(1.08)^t\,dt &= \left[\frac{2.88}{\log(1.08)}t(1.08)^t\right]_0^{\frac{5}{6}} - \int_0^{\frac{5}{6}}\frac{2.88}{\log(1.08)}t(1.08)^t\,dt\\ &= \frac{2.88}{\log(1.08)}\frac{5}{6}(1.08)^{\frac{5}{6}} - \frac{2.88}{\log(1.08)^2}\left((1.08)^{\frac{5}{6}}-1\right)\\ &= 1.0438022666\end{aligned}$$

Expected inflation from the start of 2025 to a random claim in Policy Year 2025 is

$$\begin{aligned}\int_0^1 t(1.08)^t\,dt + \int_1^2 (2-t)(1.08)^t\,dt &= \int_0^1 t(1.08)^t\,dt + (1.08)\int_0^1 (1-t)(1.08)^t\,dt\\ &= 1.08\int_0^1 (1.08)^t\,dt - (0.08)\int_0^1 t(1.08)^t\,dt\\ &= 1.08\frac{0.08}{\log(1.08)} - 0.08\left(\left[\frac{t(1.08)^t}{\log(1.08)}\right]_0^1 - \int_0^1\frac{(1.08)^t}{\log(1.08)}\,dt\right)\\ &= 1.08\frac{0.08}{\log(1.08)} - 0.08\left(\frac{1.08}{\log(1.08)} - \frac{0.08}{\log(1.08)^2}\right)\\ &= \frac{0.08^2}{\log(1.08)^2}\\ &= 1.08053317542\end{aligned}$$

Therefore, the premium for policy year 2025 is $644(1.08)^{2+\frac{5}{6}}\frac{1.08053317542}{1.0438022666}=\$829.10$.

10 The loss ratio is $\frac{76594400}{85346200} = 0.897455305567$. Therefore, to achieve a loss ratio of 0.8, the premium needs to be increased by a factor $\frac{0.897455305567}{0.8} = 1.12181913196$. The new premium is therefore $974 \times 1.12181913196 = \$1,092.65$.

Using the loss-cost method, the total exposure is $\frac{85346200}{974} = 87624.4353183$ units of exposure, so the expected loss per unit of exposure is $\frac{76594400}{87624.4353183} = \$874.121467622$. The premium is therefore $\frac{874.121467622}{0.8} = \$1,092.65$.

11

We first need to adjust the earned premiums to the new rate. Under uniform distribution, we have that $\frac{1}{2}\times\frac{3}{4}\times\frac{3}{4}=\frac{9}{32}$ of the policies are at rate \$432; $\frac{1}{2}\times\frac{5}{12}\times\frac{5}{12}=\frac{25}{288}$ of the policies are at rate \$491; and the remaining $\frac{182}{288}$ are at rate \$464. The average premium per policy is therefore $\frac{9}{32}\times 432+\frac{182}{288}\times 464+\frac{25}{288}\times 491=457.34375$. The adjusted earned premiums are therefore $1700000\times\frac{491}{457.34375}=1825104.20226$. The loss ratio at this premium is therefore $\frac{1520000}{1825104.20226}=0.832829160175$. The new premium before inflation is therefore $491\times\frac{0.832829160175}{0.8}=511.148897058$.

To calculate inflation from accident year 2018 to policy year 2020, we calculate $\int_0^1(1.04)^t dt=\left[\frac{(1.04)^t}{\log(1.04)}\right]_0^1=\frac{0.04}{\log(1.04)}=1.01986926764$ and

$$\begin{aligned}\int_0^1 t(1.04)^t\,dt+\int_1^2(2-t)(1.04)^t\,dt &= \int_0^1 t(1.04)^t\,dt+1.04\int_0^1(1-t)(1.04)^t\,dt\\ &=1.04\int_0^1(1.04)^t\,dt-0.04\int_0^1 t(1.04)^t\,dt\\ &=1.04\times 1.01986926764-0.04\left(\left[\frac{t(1.04)^t}{\log(1.04)}\right]_0^1-\int_0^1\frac{(1.04)^t}{\log(1.04)}\,dt\right)\\ &=1.06066403835-\frac{0.04\times 1.04}{\log(1.04)}+\frac{0.04\times 1.01986926764}{\log(1.04)}\\ &=1.04013332308\end{aligned}$$

The inflation is therefore $\frac{1.04^2\times 1.04013332308}{1.01986926764}=1.10309059988$, so the new premium is $1.10309059988\times 511.148897058=\$563.84$

12 The proportion of earned premiums under the new premium is $\frac{1}{2}\left(1-\frac{190}{365}\right)^2 = 0.114937136423$. Therefore, the earned premiums adjusted to the new premium are

$$3679710 \times \frac{660}{0.885062863577 \times 629 + 0.114937136423 \times 660} = \$3839314.6664$$

The loss ratio is therefore $\frac{3244610}{3839314.6664} = 0.845101348008$. Thus to achieve a loss ratio of 0.75, the base premium should be multiplied by $\frac{0.845101348008}{0.75} = 1.12680179734$.

Using 6% annual inflation, the expected inflation from the start of 2021 to a random loss in accident year 2021 is $\int_0^1 (1.06)^t\,dt = \frac{0.06}{\log(1.06)} = 1.02970867194$. The expected inflation from the start of 2023 to a random loss in policy year 2023 is

$$\int_0^1 t(1.06)^t\,dt + \int_1^2 (2-t)(1.06)^t\,dt = \frac{0.06^2}{\log(1.06)^2} = 1.06029994908$$

The new base premium is therefore $660 \times 1.12680179734 \times \frac{1.06^2 \times 1.06029994908}{1.02970867194} = \$860.43$.

IRLRPCI 3 Loss Reserving

3.6 Loss Reserving Methods

13

Claim Type	Policy Year	Earned Premiums	Expected Loss Ratio	Expected Claims	Losses paid to date	Reserves needed
Collision	2014	$200,000	0.79	$158,000	$130,000	$28,000
	2015	$250,000	0.79	$197,500	$110,000	$87,500
	2016	$270,000	0.77	$207,900	$60,000	$147,900
Comprehensive	2014	$50,000	0.74	$37,000	$36,600	$400
	2015	$60,000	0.72	$43,200	$44,300	$0
	2016	$65,000	0.75	$48,750	$41,400	$7,350
Bodily Injury	2014	$300,000	0.73	$219,000	$86,000	$133,000
	2015	$500,000	0.73	$365,000	$85,000	$280,000
	2016	$600,000	0.72	$432,000	$12,000	$420,000

The loss reserves needed are therefore $28000+87500+147900+400+0+7350+133000+280000+420000 = \$1,104,150$.

14

The average, 3-year average and mean loss development factors are:

Accident year	Development year 1/0	2/1	3/2	4/3	5/4
Average	1.187962	1.200218	1.11665	1.052196	1.010355
3-year average	1.143815	1.117381	1.11665	1.052196	1.010355
Mean	$\frac{47549}{40254} = 1.181224$	$\frac{43782}{37279} = 1.174441$	$\frac{36398}{32559} = 1.117909$	$\frac{23967}{22966} = 1.043586$	$\frac{11709}{11589} = 1.010355$

The estimated future cumulative payments are then calculated by multiplying the most recent cumulative payment by the corresponding loss development factors. The three methods result in the following estimated cumulative payments:

Average:

Accident year	Development year 0	1	2	3	4	5
2012					12378	12506
2013				13432	14133	14279
2014			11223	12532	13186	13323
2015		10270	12326	13764	14483	14632
2016	11290	13412	16097	17975	18913	19109

3-year average:

Accident year	Development year 0	1	2	3	4	5
2012					12378	12506
2013				13432	14133	14279
2014			11223	12532	13186	13323
2015		10270	11476	12814	13483	13623
2016	11290	12914	14429	16113	16954	17129

mean:

Accident year	Development year 0	1	2	3	4	5
2012					12378	12506
2013				13432	14017	14163
2014			11223	12546	13093	13229
2015		10270	12062	13484	14071	14217
2016	11290	13336	15662	17509	18272	18461

15

First we calculate the expected Loss payments. Using the loss development factors, the proportion of payments made in each year is:

Cumulative	0.5889500	0.6996502	0.8397328	0.9376876	0.9866311	1
Proportion	0.5889500	0.1107002	0.1400826	0.0979548	0.0489435	0.0133689

This leads to expected payments:

Policy	Expected	Development Year					
Year	loss	0	1	2	3	4	5
2012	129,600	76,327.92	14,346.75	18,154.70	12,694.95	6,343.08	1,732.60
2013	147,600	86,929.02	16,339.35	20,676.19	14,458.13	7,224.07	1,973.24
2014	151,200	89,049.24	16,737.87	21,180.48	14,810.77	7,400.26	2,021.37
2015	158,400	93,289.68	17,534.91	22,189.08	15,516.04	7,752.66	2,117.63
2016	194,400	114,491.88	21,520.12	27,232.05	19,042.42	9,514.62	2,598.90

2 Random Variables

Key Functions

16

$$S_1(x) = 1 - F_1(x) = 1 - \begin{cases} 0 & x < 0 \\ 0.01x & 0 \leqslant x < 100 \\ 1 & x \geqslant 100 \end{cases} = \begin{cases} 1 & x < 0 \\ 1 - 0.01x & 0 \leqslant x < 100 \\ 0 & x \geqslant 100 \end{cases}$$

$$f_1(x) = \frac{d}{dx} F_1(x) = \begin{cases} 0.01 & 0 \leqslant x < 100 \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda_1(x) = \frac{f_1(x)}{S_1(x)} = \frac{0.01}{1 - 0.01x} = \frac{1}{100 - x}$$

for $0 < x < 100$.

17

$$S_2(x) = 1 - F_2(x) = 1 - \begin{cases} 0 & x < 0 \\ 1 - \left(\frac{2000}{x+2000}\right)^3 & x \geqslant 0 \end{cases} = 1 - \begin{cases} 1 & x < 0 \\ \left(\frac{2000}{x+2000}\right)^3 & x \geqslant 0 \end{cases}$$

$$f_2(x) = \frac{d}{dx} F_2(x) = \begin{cases} 0 & x < 0 \\ \frac{2000^3}{2(x+2000)^2} & x \geqslant 0 \end{cases}$$

$$\lambda_2(x) = \frac{f_2(x)}{S_2(x)} = \begin{cases} 0 & x < 0 \\ \frac{\left(\frac{2000^3}{2(x+2000)^2}\right)}{\left(\frac{2000^3}{(x+2000)^3}\right)} & x \geqslant 0 \end{cases} = \begin{cases} 0 & x < 0 \\ \frac{x+2000}{2} & x \geqslant 0 \end{cases}$$

18

$$S_3(x) = 1 - F_3(x) = 1 - \begin{cases} 0 & x < 0 \\ 0.5 & 0 \leqslant x < 1 \\ 0.75 & 1 \leqslant x < 2 \\ 0.87 & 2 \leqslant x < 3 \\ 0.95 & 3 \leqslant x < 4 \\ 1 & x \geqslant 4 \end{cases} = \begin{cases} 1 & x < 0 \\ 0.5 & 0 \leqslant x < 1 \\ 0.25 & 1 \leqslant x < 2 \\ 0.13 & 2 \leqslant x < 3 \\ 0.05 & 3 \leqslant x < 4 \\ 0 & x \geqslant 4 \end{cases}$$

$$f_3(x) = \begin{cases} 0.5 & x = 0 \\ 0.25 & x = 1 \\ 0.12 & x = 2 \\ 0.08 & x = 3 \\ 0.05 & x = 4 \end{cases}$$

19

$$S_4(x) = 1 - F_4(x) = 1 - \begin{cases} 0 & x < 0 \\ 1 - 0.3e^{-0.00001x} & x \geqslant 0 \end{cases} = \begin{cases} 1 & x < 0 \\ 0.3e^{-0.00001x} & x \geqslant 0 \end{cases}$$

$$f_4(x) = \frac{d}{dx}F_4(x) = \begin{cases} 1 & x < 0 \\ 0.000003e^{-0.00001x} & x \geqslant 0 \end{cases}$$

$$\lambda_4(x) = \frac{f_4(x)}{S_4(x)} = \begin{cases} 1 & x < 0 \\ \frac{0.000003e^{-0.00001x}}{0.3e^{-0.00001x}} & x \geqslant 0 \end{cases} = \begin{cases} 1 & x < 0 \\ 0.00001 & x \geqslant 0 \end{cases}$$

3.1 Moments

20

$$f_1(x) = 0.01 \qquad\qquad \text{for } 0 < x < 100$$

$$\begin{aligned}
\int_0^{100} 0.01x\,dx &= 0.01\frac{100^2}{2} = 50\\
\int_0^{100} 0.01(x-50)^2\,dx &= 0.01\int_{-50}^{50} u^2\,du = 0.01\times 2\frac{50^3}{3} = \frac{2500}{3}\\
\int_0^{100} 0.01(x-50)^3\,dx &= 0.01\int_{-50}^{50} u^3\,du = 0.01\frac{50^4-(-50)^4}{4} = 0\\
\int_0^{100} 0.01(x-50)^4\,dx &= 0.01\int_{-50}^{50} u^4\,du = 0.02\frac{50^5}{5} = 1250000
\end{aligned}$$

Coefficient of variation	$\frac{\left(\frac{50}{\sqrt{3}}\right)}{50} = \frac{1}{\sqrt{3}}$
Skewness	$\frac{0}{\left(\frac{2500}{3}\right)^{\frac{3}{2}}} = 0$
Kurtosis	$\frac{1250000}{\left(\frac{2500}{3}\right)^2} = 1.8$

$$F_2(x) = 1 - \left(\frac{2000}{2000+x}\right)^3 \qquad\qquad \text{for } 0 < x$$

$$\int_0^\infty \left(\frac{2000}{2000+x}\right)^3 dx = 2000\int_1^\infty u^{-3}\,du$$
$$= 2000\left[-\frac{u^{-2}}{2}\right]_1^\infty = 1000$$
$$\int_0^\infty 2x\left(\frac{2000}{2000+x}\right)^3 dx = \int_0^\infty 2(2000+x)\left(\frac{2000}{2000+x}\right)^3 dx - 4000\int_0^\infty \left(\frac{2000}{2000+x}\right)^3 dx$$
$$= 2000\left(4000\int_1^\infty u^{-2}\,du - 4000\int_1^\infty u^{-2}\,du\right)$$
$$= 2000\left(4000\left[-\frac{1}{u}\right]_1^\infty - 4000\left[\frac{u^{-2}}{2}\right]_1^\infty u^{-2}\,du\right)$$
$$= 2000(4000-2000) = 4000000$$
$$\int_0^\infty 3x^2\left(\frac{2000}{2000+x}\right)^3 dx = \int_0^\infty 3(2000+x)^2\left(\frac{2000}{2000+x}\right)^3 dx - 12000\int_0^\infty (2000+x)\left(\frac{2000}{2000+x}\right)^3 dx$$
$$+ 12000000\int_0^\infty \left(\frac{2000}{2000+x}\right)^3 dx$$
$$= 2000\left(12000000\int_1^\infty u^{-1}\,du - 24000000\int_1^\infty u^{-2}\,du + 12000000\int_1^\infty u^{-3}\,du\right)$$
$$= \infty$$

$\mu_2 = \mu_2' - (\mu_1)^2 = 4000000 - 1000^2 = 3000000$

Coefficient of variation	$\frac{\sqrt{3000000}}{1000} = \sqrt{3}$
Skewness	undefined
Kurtosis	undefined

$$f_3(x) = \begin{cases} 0.5 & x=0 \\ 0.25 & x=1 \\ 0.12 & x=2 \\ 0.08 & x=3 \\ 0.05 & x=4 \end{cases}$$

x	x^2	x^3	x^4	$P(X=x)$	xp	x^2p	x^3p	x^4p
0	0	0	0	0.5	0	0	0	0
1	1	1	1	0.25	0.25	0.25	0.25	0.25
2	4	8	16	0.12	0.24	0.48	0.96	1.92
3	9	27	81	0.08	0.24	0.72	2.16	6.48
4	16	64	256	0.05	0.2	0.8	3.2	12.8
					$E(X) = 0.93$	$E(X^2) = 2.25$	$E(X^3) = 6.57$	$E(X^4) = 21.45$

μ_2	$2.25 - 0.93^2 = 1.3851$
μ_3	$6.57 - 3 \times 0.93 \times 2.25 + 2 \times 0.93^3 = 1.901214$
μ_4	$21.45 - 4 \times 0.93 \times 6.57 + 6 \times 0.93^2 \times 2.25 - 3 \times 0.93^4 = 6.44159397$

Coefficient of variation	$\frac{\sqrt{1.3851}}{0.93} = 1.26548679006$
Skewness	$\frac{1.901214}{1.3851^{\frac{3}{2}}} = 1.16629740612$
Kurtosis	$\frac{6.44159397}{1.3851^2} = 3.35761648225$

$$F_4(x) = \begin{cases} 0 & x < 0 \\ 1 - 0.3e^{-0.00001x} & x \geqslant 0 \end{cases}$$

$$\begin{aligned}
E(X) &= \int_0^\infty 0.3e^{-0.00001x}\,dx = 30000 \\
E(X^2) &= \int_0^\infty 0.6xe^{-0.00001x}\,dx = [-60000xe^{-0.00001x}]_0^\infty + 60000\int_0^\infty e^{-0.00001x}\,dx = 6 \times 10^9 \\
E(X^3) &= \int_0^\infty 0.9x^2e^{-0.00001x}\,dx = \int_0^\infty 90000xe^{-0.00001x}\,dx = 1.8 \times 10^{15} \\
E(X^4) &= \int_0^\infty 1.2x^3e^{-0.00001x}\,dx = \int_0^\infty 360000x^2e^{-0.00001x}\,dx = 7.2 \times 10^{20}
\end{aligned}$$

μ_2	$6 \times 10^9 - 30000^2 = 5.1 \times 10^9$
μ_3	$1.8 \times 10^{15} - 3 \times 30000 \times 6 \times 10^9 + 2 \times 30000^3 = 1.314 \times 10^{15}$
μ_4	$7.2 \times 10^{20} - 4 \times 30000 \times 1.8 \times 10^{15} + 6 \times 30000^2 \times 6 \times 10^9 - 3 \times 30000^4 = 5.3397 \times 10^{20}$

Coefficient of variation	$\frac{\sqrt{5.1\times10^9}}{30000} = 2.38047614285$
Skewness	$\frac{1.314\times10^{15}}{(5.1\times10^9)^{\frac{3}{2}}} = 3.6077804518$
Kurtosis	$\frac{5.3397\times10^{20}}{(5.1\times10^9)^2} = 20.5294117647$

21

$$\mathbb{E}(X) = \int_0^\infty \frac{x^\alpha e^{-\frac{x}{\theta}}}{\theta^\alpha \Gamma(\alpha)}\, dx \qquad = \theta \int_0^\infty \frac{u^\alpha e^{-u}}{\Gamma(\alpha)}\, du$$

$$= \theta \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} = \alpha\theta$$

$$\mathbb{E}(X^2) = \theta^2 \int_0^\infty \frac{u^{\alpha+2} e^{-u}}{\Gamma(\alpha)}\, du$$

$$= \theta^2 \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} = \alpha(\alpha+1)\theta^2$$

$$\mathbb{E}(X^n) = \theta^n \int_0^\infty \frac{u^{\alpha+n} e^{-u}}{\Gamma(\alpha)}\, du$$

$$= \theta^n \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \alpha(\alpha+1)\cdots(\alpha+n-1)\theta^n$$

Calculating the centralised moments,

$$\mu_2 = \mu_2' - (\mu_1)^2 = \alpha(\alpha+1)\theta^2 - (\alpha\theta)^2 = \alpha\theta^2$$

$$\mu_3 = \mu_3' - 3\mu_1\mu_2' + 2(\mu_1)^3 = \left(\alpha(\alpha+1)(\alpha+2) - 3\alpha^2(\alpha+1) + 2\alpha^3\right)\theta^3 = 2\alpha\theta^3$$

$$\mu_4 = \left(\alpha(\alpha+1)(\alpha+2)(\alpha+3) - 4\alpha^2(\alpha+1)(\alpha+2) + 6\alpha^3(\alpha+1) - 3\alpha^4\right)\theta^4 = (3\alpha^2 - 2\alpha)\theta^4$$

Coefficient of variation	$\frac{\sqrt{\alpha\theta^2}}{\theta} = \sqrt{\alpha}$
Skewness	$\frac{2\alpha\theta^3}{(\alpha\theta^2)^{\frac{3}{2}}} = \frac{2}{\sqrt{\alpha}}$
Kurtosis	$\frac{(3\alpha^2-2\alpha)\theta^4}{(\alpha\theta^2)^2} = 3 - \frac{2}{\alpha^2}$

22 The mean excess loss function is the integral of the survival function.

$$l(x) = \mathbb{E}((X-x)_+) = \int_x^\infty S(t)\,dt$$

For

$$F_2(x) = \begin{cases} 0 & x < 0 \\ 1 - \left(\frac{2000}{x+2000}\right)^3 & x \geqslant 0 \end{cases}$$

This becomes

$$\begin{aligned} l(x) &= \int_x^\infty \left(\frac{2000}{t+2000}\right)^3 dt \\ &= 2000 \int_{\frac{x+2000}{2000}}^\infty u^{-3}\,du \\ &= 2000 \left[-\frac{1}{2u^2}\right]_{\frac{x+2000}{2000}}^\infty \\ &= \frac{2000}{2\left(\frac{x+2000}{2000}\right)^2} \\ &= \frac{2000^3}{2(x+2000)^2} \end{aligned}$$

23 The density function is $\frac{xe^{-\frac{x}{\theta}}}{\theta^2}$, so the median is the solution to

$$\int_m^\infty \frac{xe^{-\frac{x}{\theta}}}{\theta^2}\,dx = \frac{1}{2}$$

or

$$\int_m^\infty xe^{-\frac{x}{\theta}}\,dx = \frac{\theta^2}{2}$$

Integrating by parts gives

$$\begin{aligned}\int_m^\infty xe^{-\frac{x}{\theta}}\,dx &= \left[-\theta xe^{-\frac{x}{\theta}}\right]_m^\infty + \int_m^\infty \theta xe^{-\frac{x}{\theta}}\,dx \\ &= \theta me^{-\frac{m}{\theta}} + \theta^2 e^{-\frac{m}{\theta}}\end{aligned}$$

So the median is the solution to

$$\begin{aligned}\theta me^{-\frac{m}{\theta}} + \theta^2 e^{-\frac{m}{\theta}} &= \frac{\theta^2}{2} \\ \left(1+\frac{m}{\theta}\right)e^{-\frac{m}{\theta}} &= \frac{1}{2}\end{aligned}$$

Numerically we obtain $m = 1.6783470\theta$. For $\theta = \frac{e^2}{6}$, this gives $m = 2.0669$.

24 The 100pth percentile of the excess loss random variable $(X-d)_+$ is $(\pi_p - d)_+$. The 100pth percentile of the limited loss random variable $X \wedge u$ is $\begin{cases} \pi_p & \text{if } \pi_p < u \\ u & \text{if } \pi_p > u \end{cases}$

25 We have that $f(x) \propto x(1-x)^{\beta-1}$. To have the 95th percentile equal to 0.8, we need

$$\begin{aligned}
P(X \leqslant 0.8) &= 0.95 \\
\frac{\int_0^{0.8} x(1-x)^{\beta-1}\,dx}{\int_0^1 x(1-x)^{\beta-1}\,dx} &= 0.95 \\
\int_0^{0.8} x(1-x)^{\beta-1}\,dx &= 0.95 \int_0^1 x(1-x)^{\beta-1}\,dx \\
\int_{0.2}^1 (1-u)u^{\beta-1}\,du &= 0.95 \int_0^1 (1-u)u^{\beta-1}\,du \\
\int_0^{0.2} u^{\beta-1} - u^\beta\,du &= 0.05 \int_0^1 u^{\beta-1} - u^\beta\,du \\
\left[\frac{u^\beta}{\beta} - \frac{u^{\beta+1}}{\beta+1}\right]_0^{0.2} &= 0.05\left[\frac{u^\beta}{\beta} - \frac{u^{\beta+1}}{\beta+1}\right]_0^1 \\
\frac{0.2^\beta}{\beta} - \frac{0.2^{\beta+1}}{\beta+1} &= 0.05\left(\frac{1}{\beta} - \frac{1}{\beta+1}\right) \\
0.2^\beta(\beta+1-0.2\beta) &= 0.05
\end{aligned}$$

Numerically, the solution to this is $\beta = 2.5526167478$.

3.3 Generating Functions and Sums of Random Variables

26 The moment generating function of a gamma distribution is

$$\begin{aligned}
\mathbb{E}(e^{Xt}) &= \int_0^\infty \frac{x^{\alpha-x}e^{-\frac{x}{\theta}}}{\theta^\alpha\Gamma(\alpha)}e^{xt}\,dx \\
&= \int_0^\infty \frac{x^{\alpha-x}e^{x\left(t-\frac{1}{\theta}\right)}}{\theta^\alpha\Gamma(\alpha)}\,dx \\
&= \frac{\left(\frac{\theta}{t\theta-1}\right)^\alpha}{\theta^\alpha}\int_0^\infty \frac{x^{\alpha-x}e^{\frac{x}{\left(\frac{\theta}{t\theta-1}\right)}}\left(\frac{\theta}{t\theta-1}\right)^\alpha\Gamma(\alpha)}{}dx \\
&= \frac{1}{(1-t\theta)^\alpha}
\end{aligned}$$

Therefore a sum of two independent gamma distributions with parameters α and θ and α' and θ (same value of θ) has moment generating function

$$M_{X_1+X_2}(t) = M_{X_1}(t)M_{X_2}(t) = (1-t\theta)^{-\alpha}(1-t\theta)^{-\alpha'} = (1-t\theta)^{-(\alpha+\alpha')}$$

which is the moment generating function of a gamma distribution with the same θ and the sum of the α values. In particular, the sum of 16 i.i.d. gamma random variables with $\alpha = 1$, $\theta = 250$ is a gamma random variable with $\theta = 250$ and $\alpha = 16$.

The probability that this aggregate exceeds 6000 is therefore the probability that a gamma distribution with $\alpha = 16$ and $\theta = 250$ exceeds 6000, which is 0.03440009.

27
(a) The probability generating function of a negative binomial distribution is

$$\begin{aligned}
P_N(z) &= \mathbb{E}(z^N) \\
&= \sum_{n=0}^{\infty} \binom{r+n-1}{n} \beta^n (1+\beta)^{-(r+n)} z^n \\
&= \sum_{n=0}^{\infty} \binom{r+n-1}{n} (z\beta)^n (1+\beta)^{-(r+n)} \\
&= (1+\beta)^{-r} \sum_{n=0}^{\infty} \binom{r+n-1}{n} \left(\frac{z\beta}{1+\beta}\right)^n \\
&= (1+\beta)^{-r} \left(1 - \frac{z\beta}{1+\beta}\right)^{-r} \\
&= (1+\beta-z\beta)^{-r}
\end{aligned}$$

(b) The sum of negative binomial distributions with the same β and r_1 and r_2 has probability generating function

$$P_N(z) = (1+\beta-z\beta)^{-r_1}(1+\beta-z\beta)^{-r_2} = (1+\beta-z\beta)^{-(r_1+r_2)}$$

is a negative binomial distribution with parameters β and $r_1 + r_2$.

28 The survival function is $\left(\frac{\theta}{\theta+x}\right)^\alpha$. The kth moment is therefore

$$\int_0^\infty kx^{k-1}\left(\frac{\theta}{\theta+x}\right)^\alpha\,dx$$

Substituting $u=\frac{\theta+x}{\theta}$ this becomes

$$\theta\int_0^\infty k(\theta(u-1))^{k-1}u^{-\alpha}\,du = k\theta^k\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}\int_1^\infty u^{i-\alpha}\,du$$

The integrals $\int_1^\infty u^{i-\alpha}\,du$ clearly exist whenever $i-\alpha<-1$, which happens for all $i\leqslant k-1$ if and only if $k<i$.

29 The mean of the Pareto distribution is $\frac{\theta}{\alpha-1}$, so we have $\frac{\theta}{\alpha-1} = \frac{\theta'}{\alpha'-1}$, or $\theta' = \theta\frac{\alpha'-1}{\alpha-1}$.
From 28, we have that the kth moment of a Pareto distribution with parameters α and θ is

$$\begin{aligned}
k\theta^k \sum_{i=0}^{k-1} \binom{k-1}{i} \int_1^\infty u^{i-\alpha}\,du &= k\theta^k \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} \frac{1}{\alpha-i-1} \\
&= k\theta^k \left(\frac{1}{\alpha-1} + \frac{(-1)^{k-1}}{\alpha-k} + \sum_{i=1}^{k-2} (-1)^i \left(\binom{k-2}{i} + \binom{k-2}{i-1} \right) \frac{1}{\alpha-i-1} \right) \\
&= k\theta^k \left(\sum_{i=1}^{k-2} (-1)^i \binom{k-2}{i} \frac{1}{\alpha-i-1} + \sum_{i=1}^{k-1} (-1)^i \binom{k-2}{i-1} \frac{1}{\alpha-i-1} \right) \\
&= \frac{k\theta}{k-1}\mathbb{E}(X^{k-1}) - \frac{k\theta}{k-1}\mathbb{E}(\tilde{X}^{k-1})
\end{aligned}$$

where $\tilde{X}$ follows a Pareto distribution with parameters $\alpha - 1$ and θ. By induction, we can therefore show that

$$\mathbb{E}(X^k) = \frac{k!\theta^k}{(\alpha-1)\cdots(\alpha-k)}$$

We therefore need to show that

$$\frac{k!\theta^k}{(\alpha-1)\cdots(\alpha-k)} > \frac{k!(\theta')^k}{(\alpha'-1)\cdots(\alpha'-k)}$$

Since $\frac{\theta}{\theta'} = \frac{\alpha-1}{\alpha'-1}$, it is equivalent to show that

$$\begin{aligned}
\frac{(\alpha-1)^k}{(\alpha-1)\cdots(\alpha-k)} &> \frac{(\alpha'-1)^k}{(\alpha'-1)\cdots(\alpha'-k)} \\
\frac{(\alpha-1)\cdots(\alpha-k)}{(\alpha-1)^k} &< \frac{(\alpha'-1)\cdots(\alpha'-k)}{(\alpha'-1)^k} \\
\left(1-\frac{1}{\alpha-1}\right)\cdots\left(1-\frac{k-1}{\alpha-1}\right) &< \left(1-\frac{1}{\alpha'-1}\right)\cdots\left(1-\frac{k-1}{\alpha'-1}\right)
\end{aligned}$$

This holds because all terms are positive, and each term on the left is smaller than the corresponding term on the right.

30 $f_Y(x) = \frac{x^{\alpha-1}e^{-\frac{x}{\theta}}}{\theta^\alpha \Gamma(\alpha)}$ and $f_x(x) = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}$, so

$$\frac{f_Y(x)}{f_x(x)} = \frac{\sqrt{2\pi}\sigma x^{\alpha-1}e^{-\frac{x}{\theta}}}{\theta^\alpha \Gamma(\alpha) e^{-\frac{(x-\mu)^2}{2\sigma^2}}} = \frac{\sqrt{2\pi}\sigma x^{\alpha-1}}{\theta^\alpha \Gamma(\alpha)} e^{\frac{(x-\mu)^2}{2\sigma^2} - \frac{x}{\theta}}$$

Since $\frac{(x-\mu)^2}{2\sigma^2} - \frac{x}{\theta} \to \infty$ as $x \to \infty$, we have that $\frac{f_Y(x)}{f_x(x)} \to \infty$, so the Gamma distribution has the heavier tail.

31 $f_Y(x) = \frac{\alpha\theta'^\alpha}{(x+\theta')^{\alpha+1}}$ and $f_x(x) = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}$, so

$$\frac{f_Y(x)}{f_x(x)} = \frac{\theta'^\alpha (x+\theta)^{\alpha+1}}{\theta^\alpha (x+\theta')^{\alpha+1}} = \frac{x+\theta}{x+\theta'} \left(\frac{x\theta' + \theta\theta'}{x\theta + \theta\theta'} \right)^\alpha$$

As $x \to \infty$, we have that $\frac{x+\theta}{x+\theta'} \to 1$ and $\frac{x\theta'+\theta\theta'}{x\theta+\theta\theta'} \to 1$, so $\frac{f_Y(x)}{f_x(x)} \to 1$, so neither distribution has the heavier tail.

32 For the Pareto distribution, $f(x) = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}$ and $S(x) = \frac{\theta^\alpha}{(x+\theta)^\alpha}$, so the hazard rate is

$$\lambda(x) = \frac{f(x)}{S(x)} = \frac{\alpha}{(x+\theta)} \to 0$$

For the Gamma distribution, let $h(v) = S\left(\frac{1}{v}\right)$ and $g(v) = f\left(\frac{1}{v}\right)$. We want to find

$$\lim_{x\to\infty} \frac{f(x)}{S(x)} = \lim_{v\to 0} \frac{g(v)}{h(v)} = \lim_{v\to 0} \frac{g'(v)}{h'(v)} = \lim_{v\to 0} \frac{f'(x)\frac{dx}{dv}}{S'(x)\frac{dx}{dv}} = \lim_{x\to\infty} \frac{f'(x)}{S'(x)} = \lim_{x\to\infty} \frac{-f'(x)}{f(x)} = -\lim_{x\to\infty} \frac{d\log(f(x))}{dx}$$

For the Gamma distribution, $f(x) = \frac{x^{\alpha-1}e^{-\frac{x}{\theta}}}{\theta^\alpha\Gamma(\alpha)}$, so $\log(f(x)) = (\alpha-1)\log(x) - \frac{x}{\theta} - \alpha\log(\theta) - \log(\Gamma(\alpha))$, and $-\frac{d\log(f(x))}{dx} = \frac{1}{\theta} - \frac{\alpha-1}{x} \to \frac{1}{\theta}$ as $x \to \infty$. Thus the hazard rate of a Gamma distribution converges to a non-zero constant.

33 The mean excess loss function is the integral of the survival function.

$$l(x) = \mathbb{E}((X-x)_+) = \int_x^\infty S(t)\,dt$$

For a Pareto distribution

$$S(x) = \left(\frac{\theta}{x+\theta}\right)^\alpha$$

so

$$\begin{aligned}
l(x) &= \int_x^\infty \left(\frac{\theta}{t+\theta}\right)^\alpha dt \\
&= \theta \int_{\frac{x+\theta}{\theta}}^\infty u^{-\alpha}\,du \\
&= \theta \left[-\frac{1}{(\alpha-1)u^{\alpha-1}}\right]_{\frac{x+\theta}{\theta}}^\infty \\
&= \frac{\theta}{(\alpha-1)\left(\frac{x+\theta}{\theta}\right)^{\alpha-1}} \\
&= \frac{\theta^\alpha}{(\alpha-1)(x+\theta)^{\alpha-1}}
\end{aligned}$$

Subadditivity Let X and Y be random variables. We have $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$ and

$$\operatorname{Var}(X+Y) = \operatorname{Var}(X)+\operatorname{Var}(Y)+2\operatorname{Cov}(X,Y) \leqslant \operatorname{Var}(X)+\operatorname{Var}(Y)+2\sqrt{\operatorname{Var}(X)\operatorname{Var}(Y)} = \left(\sqrt{\operatorname{Var}(X)}+\sqrt{\operatorname{Var}(Y)}\right)^2$$

Therefore, we have $\sqrt{\operatorname{Var}(X+Y)} \leqslant \sqrt{\operatorname{Var}(X)} + \sqrt{\operatorname{Var}(Y)}$, so

$$\mathbb{E}(X+Y) + k\sqrt{\operatorname{Var}(X+Y)} \leqslant \mathbb{E}(X) + \mathbb{E}(Y) + k\sqrt{\operatorname{Var}(X)+\operatorname{Var}(Y)}$$

Monotonicity This is not necessarily true. For example, if Y is a Bernoulli random variable with probability p, and $X = Y + \epsilon(1-Y)$, for some $\epsilon > 0$, then clearly $X \geqslant Y$ with probability 1, and $E(Y) = p$, $E(X) = p + \epsilon(1-p)$, $\operatorname{Var}(Y) = p(1-p)$ and $\operatorname{Var}(X) = (1-\epsilon)^2 p(1-p)$, so for the standard deviation principle to be monotonic, we must have

$$\begin{aligned} p + \epsilon(1-p) + k(1-\epsilon)\sqrt{p(1-p)} &\geqslant p + k\sqrt{p(1-p)} \\ \epsilon(1-p-k\sqrt{p(1-p)} &\geqslant 0 \\ k &\leqslant \frac{1-p}{\sqrt{p(1-p)}} = \sqrt{\frac{1-p}{p}} \end{aligned}$$

Thus for any $k > 0$, we can choose a p such that the standard deviation principle is not monotonic.

Positive Homogeneity We have

$$\rho(aX) = \mathbb{E}(aX) + k\sqrt{\operatorname{Var}(aX)} = a\mathbb{E}(X) + k\sqrt{a^2\operatorname{Var}(X)} = a\rho(X)$$

Translation Invariance We have

$$\rho(X+c) = \mathbb{E}(X+c) + k\sqrt{\operatorname{Var}(X+c)} = \mathbb{E}(X) + c + k\sqrt{\operatorname{Var}(X)} = \rho(X) + c$$

35

$$F_X(x) = \left(\frac{x}{x+\theta}\right)^\tau$$

$\mathrm{VaR}_p(X)$ is the solution to

$$\begin{aligned}\left(\frac{x}{x+\theta}\right)^\tau &= p\\ \frac{x+\theta}{x} &= p^{-\frac{1}{\tau}}\\ \frac{\theta}{x} &= p^{-\frac{1}{\tau}} - 1\\ x &= \frac{\theta}{p^{-\frac{1}{\tau}} - 1}\end{aligned}$$

36 The TVaR of the Pareto distribution is given by

$$
\begin{aligned}
\mathrm{TVaR}_p(X) &= \int_{\mathrm{VaR}_p(X)} S(x)\,dx \\
&= \int_{\theta\left((1-p)^{-\frac{1}{\alpha}}-1\right)} \left(\frac{\theta}{x+\theta}\right)^{\alpha} dx \\
&= \int_{\theta\left((1-p)^{-\frac{1}{\alpha}}\right)} \theta^{\alpha} u^{-\alpha}\,du \\
&= \theta^{\alpha}\left[-\frac{u^{1-\alpha}}{\alpha-1}\right]_{\theta\left((1-p)^{-\frac{1}{\alpha}}\right)} \\
&= \theta^{\alpha}\frac{\left(\theta\left((1-p)^{-\frac{1}{\alpha}}\right)\right)^{1-\alpha}}{\alpha-1} \\
&= \theta\frac{(1-p)^{-\frac{1-\alpha}{\alpha}}}{\alpha-1}
\end{aligned}
$$

37

Monotonicity If $P(X \geqslant Y) = 1$, then for all x, we have $S_X(x) \geqslant S_Y(x)$. In particular $\mathrm{VaR}_p(X) \geqslant \mathrm{Var}_p(Y)$. Now we want to show that

$$\mathrm{Var}_p(X) + \frac{1}{p}\int_{\mathrm{VaR}_p(X)}^{\infty} S_X(x)\,dx \geqslant \mathrm{Var}_p(Y) + \frac{1}{p}\int_{\mathrm{VaR}_p(Y)}^{\infty} S_Y(x)\,dx$$

Equivalently, we need to show that

$$\begin{aligned}\mathrm{Var}_p(X) + \frac{1}{1-p}\int_{\mathrm{VaR}_p(X)}^{\infty} S_X(x) - S_Y(x)\,dx &\geqslant \mathrm{Var}_p(Y) + \frac{1}{1-p}\int_{\mathrm{VaR}_p(Y)}^{\mathrm{VaR}_p(X)} S_Y(x)\,dx \\ \int_{\mathrm{VaR}_p(X)}^{\infty} S_X(x) - S_Y(x)\,dx &\geqslant \int_{\mathrm{VaR}_p(Y)}^{\mathrm{VaR}_p(X)} S_Y(x)\,dx - (1-p)(\mathrm{Var}_p(X) - \mathrm{Var}_p(Y))\end{aligned}$$

Now since $S_X(x) \geqslant S_Y(x)$, the left-hand side is positive. On the other hand, the right-hand side is $\int_{\mathrm{VaR}_p(Y)}^{\mathrm{VaR}_p(X)} (S_Y(x) - (1-p))\,dx$, and since $S_Y(\mathrm{VaR}_p(Y)) = 1-p$, the integrand is non-positive, so the right-hand side is $\leqslant 0$ as required.

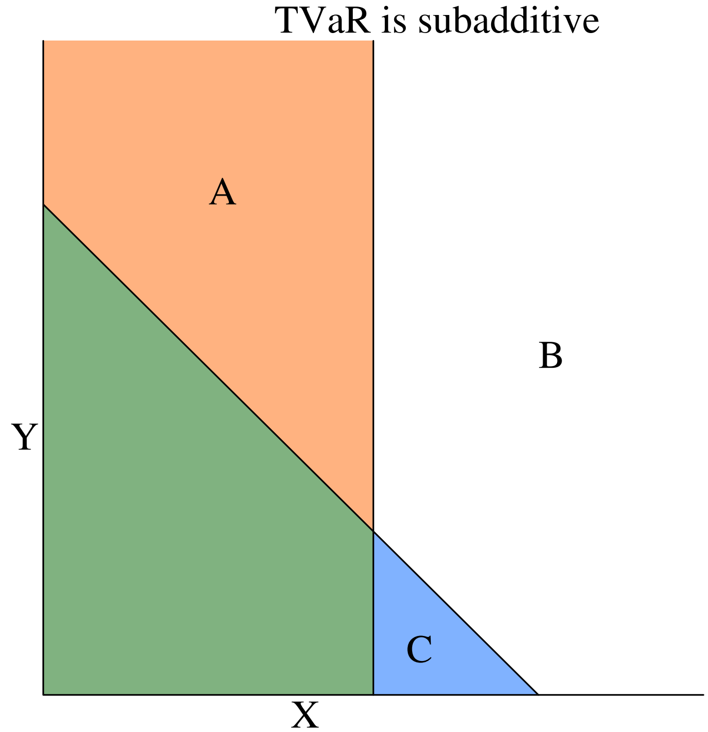
Positive Homogeneity It is obvious that $\mathrm{VaR}_p(aX) = a\,\mathrm{VaR}_p(X)$ for any $a \geqslant 0$. Therefore, $aX > \mathrm{VaR}_p(aX)$ if and only if $X > \mathrm{VaR}_p(X)$. Now

$$\begin{aligned}\mathrm{TVaR}_p(aX) &= \mathbb{E}(aX|aX > \mathrm{VaR}_p(aX)) \\ &= \mathbb{E}(aX|X > \mathrm{VaR}_p(X)) \\ &= a\mathbb{E}(X|X > \mathrm{VaR}_p(X)) \\ &= a\,\mathrm{TVaR}(X)\end{aligned}$$

Translation Invariance It is obvious that $\mathrm{VaR}_p(X+c) = \mathrm{VaR}_p(X) + c$ for any c. Therefore, $X + c > \mathrm{VaR}_p(X+c)$ if and only if $X > \mathrm{VaR}_p(X)$. Now

$$\begin{aligned}\mathrm{TVaR}_p(X+c) &= \mathbb{E}(X+c|X+c > \mathrm{VaR}_p(X+c)) \\ &= \mathbb{E}(X+c|X > \mathrm{VaR}_p(X)) \\ &= \mathbb{E}(X|X > \mathrm{VaR}_p(X)) + c \\ &= \mathrm{TVaR}(X) + c\end{aligned}$$

Subadditivity In the following diagram, the vertical line is $\mathrm{VaR}_p(X)$, while the diagonal line is $\mathrm{VaR}_p(X+Y)$.



By definition of VaR, the area to the left of the vertical line, and the area below the diagonal line both have probability p, so $P(A) = P(C)$. (A is the orange area above the diagonal line and to the left of the vertical line; C is the blue area to the right of the vertical line and below the diagonal line).

Now $\mathrm{TVaR}_p(X) = \frac{1}{1-p}\iint_{B\cup C} x f_{X,Y}(x,y)\,dx\,dy$, while $\mathrm{TVaR}_p(X+Y) = \frac{1}{1-p}\iint_{B\cup A}(x+y) f_{X,Y}(x,y)\,dx\,dy$. We can show that

$$\iint_A x f_{X,Y}(x,y)dxdy \leqslant P(A)\,\mathrm{VaR}_p(X) = P(C)\,\mathrm{VaR}_p(X) \leqslant \iint_C x f_{X,Y}(x,y)dxdy$$

Therefore,

$$\begin{aligned}\mathrm{TVaR}_p(X+Y) &= \frac{1}{1-p}\iint_{B\cup A}(x+y)f_{X,Y}(x,y)\,dx\,dy\\ &= \frac{1}{1-p}\left(\iint_{B\cup A} x f_{X,Y}(x,y)\,dx\,dy + \iint_{B\cup A} y f_{X,Y}(x,y)\,dx\,dy\right)\\ &\leqslant \frac{1}{1-p}\left(\iint_{B\cup C} x f_{X,Y}(x,y)\,dx\,dy + \iint_{B\cup A} y f_{X,Y}(x,y)\,dx\,dy\right)\\ &= \mathrm{TVaR}(X) + \frac{1}{1-p}\left(\iint_{B\cup A} y f_{X,Y}(x,y)\,dx\,dy\right)\end{aligned}$$

A similar argument shows that the second integral is at most $\mathrm{TVaR}_p(Y)$.

	x	$F(x)$	$F(\lceil x\rceil)$
	$0<x<1$	$0.16x$	0.16
	$1<x<2$	$0.34x-0.18$	0.5
38	$2<x<3$	$0.28x-0.06$	0.78
	$3<x<4$	$0.19x+0.21$	0.97
	$4<x<5$	$0.026x+0.866$	0.996
	$5<x<6$	$0.004x+0.976$	1

(a) At the 90% level, we want to solve $F(x)=0.9$, which is clearly between $x=3$ and $x=4$. In this interval, $F(x)=0.19x+0.21$, so the VaR is the solution to $0.19x+0.21=0.9$, which is $\frac{0.69}{0.19}=3.63157894737$.

Now the TVaR is

$$\begin{aligned}
\mathrm{VaR}_p(X)+10\int_{\frac{69}{19}}^{6} S(x)\,dx &= \frac{69}{19}+10\left(\int_{\frac{69}{19}}^{4}(0.79-0.19x)+\int_{4}^{5}(0.134-0.026x)\,dx+\int_{5}^{6}(0.024-0.004x)\,dx\right)\\
&= \frac{69}{19}+10\left(\left[(0.79x-0.095x^2)\right]_{\frac{69}{19}}^{4}+\left[0.134x-0.013x^2\right]_{4}^{5}+\left[0.024x-0.002x^2\right]_{5}^{6}\right)\\
&= 4.06105263161
\end{aligned}$$

(b) At the 99% level, we want to solve $F(x)=0.99$, which is clearly between $x=4$ and $x=5$. In this interval, $F(x)=0.026x+0.866$, so the VaR is the solution to $0.026x+0.866=0.99$, which is $\frac{0.124}{0.026}=\frac{62}{13}=4.76923076923$.

Now the TVaR is

$$\begin{aligned}
\mathrm{VaR}_p(X)+100\int_{\frac{62}{13}}^{6} S(x)\,dx &= \frac{62}{13}+100\left(\int_{\frac{62}{13}}^{5}(0.134-0.026x)\,dx+\int_{5}^{6}(0.024-0.004x)\,dx\right)\\
&= \frac{62}{13}+100\left(\left[0.134x-0.013x^2\right]_{\frac{67}{13}}^{5}+\left[0.024x-0.002x^2\right]_{5}^{6}\right)\\
&= 5.13076923077
\end{aligned}$$

39 The VaR can be found using the `qgamma` function in `R`

```
qgamma(0.95,shape=4,scale=2000)
```

This gives $\mathrm{VaR}_{0.95}(X) = 15507.31$.

Now to get the TVaR, we calculate

$$\begin{aligned}\mathrm{TVaR}_{0.95}(X) &= 20\int_{\mathrm{VaR}_{0.95}(X)} x\frac{x^{\alpha-1}e^{-\frac{x}{\theta}}}{\theta^{\alpha}\Gamma(\alpha)}\,dx\\ &= 20\theta\alpha\int_{\mathrm{VaR}_{0.95}(X)} \frac{x^{\alpha}e^{-\frac{x}{\theta}}}{\theta^{\alpha+1}\Gamma(\alpha+1)}\,dx\end{aligned}$$

The integral in this expression is the probability that a Gamma distribution with shape $\alpha + 1$ and scale θ exceeds $\mathrm{VaR}_{0.95}(X)$, which we can calculate using the `pgamma` function in `R`.

```
pgamma(15507.31,shape=5,scale=2000,lower.tail=FALSE)
```

This gives 0.1146317 for the integral, so

$$\begin{aligned}\mathrm{TVaR}_{0.95}(X) &= 20 \times 2000 \times 4 \times 0.1146317\\ &= 18341.072\end{aligned}$$

40

(a)

The first Pareto distribution has mean $\frac{1000}{3}$ and variance $\frac{2000000}{9}$. The second Pareto distribution has mean $\frac{1000}{7}$ and variance $\frac{4000000}{147}$.

The mean of the mixture is therefore $0.4 \times \frac{1000}{3} + 0.6 \times \frac{1000}{7} = 219.047619047$ The variance of the mixture is

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}(\mathrm{Var}(X|Z)) + \mathrm{Var}(\mathbb{E}(X|Z)) \\
&= 0.4 \times \frac{2000000}{9} + 0.6 \times \frac{4000000}{147} + \left(\frac{1000}{3} - \frac{1000}{7}\right)^2 \times 0.4 \times 0.6 \\
&= 113922.902494
\end{aligned}$$

Thus the standard deviation principle is $\rho(X) = 219.047619047 + 3.5\sqrt{113922.902494} = 1400.38396216$.

(b) The VaR at the 99% level is the solution to

$$0.4\left(\frac{1000}{1000+x}\right)^3 + 0.6\left(\frac{1000}{1000+x}\right)^7 = 0.05$$

Letting $v = \frac{1000}{1000+x}$, we get $8v^3 + 12v^7 = 1$

We solve this numerically:

```
v<-0.1
for(i in seq_len(1000)){
      v<- ((1-12*v^7)/8)^(1/3)
}
```

to get $v = 0.48671022130713$, so $x = 1000\left(\frac{1}{0.48671022130713} - 1\right) = 1054.61064145$. Now the TVaR is

$$\begin{aligned}
1054.61064145 + \int_{1054.61064145}^{\infty} S(x)\,dx &= \int_{1054.61064145}^{\infty} 0.4\left(\frac{1000}{1000+x}\right)^4 + 0.6\left(\frac{1000}{1000+x}\right)^8 dx \\
&= 1054.61064145 + \int_{2054.61064145}^{\infty} 0.4 \times 1000^4 u^{-4} + 0.6 \times 1000^8 u^{-8}\,du \\
&= 1054.61064145 + 0.4 \times \frac{1000^4}{3 \times 2054.61064145^3} + 0.6 \times \frac{1000^8}{7 \times 2054.61064145^{-7}} \\
&= 1070.5378983
\end{aligned}$$

1 4 Characteristics of Actuarial Models

4.2 The Role of Parameters

41

(a)

Exponential The Survival function $S_{cX}(x) = S_X\left(\frac{X}{c}\right)$. For the exponential distribution, this survival function is $S_X(x) = e^{-\frac{x}{\theta}}$, so

$$S_{cX}(x) = S_X\left(\frac{X}{c}\right) = e^{-\frac{x}{c\theta}}$$

which is an exponential distribution with mean $c\theta$.

Gamma The density function of a scaled random variable is given by $f_{cX}(x) = \frac{1}{c}f_X\left(\frac{x}{c}\right)$. For the Gamma distribution, this is

$$f_{cX}(x) = \frac{1}{c}\frac{\left(\frac{x}{c}\right)^{\alpha-1}e^{-\frac{x}{c\theta}}}{\theta^\alpha\Gamma(\alpha)} = \frac{x^{\alpha-1}e^{-\frac{x}{c\theta}}}{c^\alpha\theta^\alpha\Gamma(\alpha)}$$

which is the density of a Gamma distribution with shape parameter α and scale parameter θ.

Normal The density function of a scaled random variable is given by $f_{cX}(x) = \frac{1}{c}f_X\left(\frac{x}{c}\right)$. For the normal distribution, this is

$$f_{cX}(x) = \frac{1}{\sqrt{2\pi}c\sigma}e^{-\frac{\left(\frac{x}{c}-\mu\right)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}c\sigma}e^{-\frac{(x-c\mu)^2}{2c^2\sigma^2}}$$

which is the density of a normal distribution with mean $c\mu$ and variance $c^2\sigma^2$.

Pareto The Survival function $S_{cX}(x) = S_X\left(\frac{X}{c}\right)$. For the Pareto distribution, this survival function is $S_X(x) = \left(\frac{\theta}{\theta+x}\right)^\alpha = \left(\frac{1}{1+\frac{x}{\theta}}\right)^\alpha$, so

$$S_{cX}(x) = S_X\left(\frac{X}{c}\right) = \left(\frac{1}{1+\frac{x}{c\theta}}\right)^\alpha$$

which is the survival function of a Pareto distribution with parameters α and $c\theta$.

(b) Which of the above distributions have scale parameters?

For the exponential, Gamma and Pareto distributions, θ is a scale paremeter. For the normal distribution, there is no scale parameter with the usual parametrisation. [There are commonly used parametrisations for the Gamma (and the exponential) distribution that do not have a scale parameter.]

42 This year's claims follow a Pareto distribution with $\alpha = 2$ and θ. Since θ is a scale parameter, after 6% inflation, next year's claims will follow a Pareto distribution with $\alpha = 2$ and 1.06θ.

The proportion of claims this year that exceed d is $S_X(d) = \left(\frac{\theta}{\theta+d}\right)^2$, while the proportion of claims next year that exceed d is $S_{1.06X}(d) = \left(\frac{1.06\theta}{1.06\theta+d}\right)^2$. The ratio r is therefore

$$\frac{\left(\frac{1.06\theta}{1.06\theta+d}\right)^2}{\left(\frac{\theta}{\theta+d}\right)^2} = \left(\frac{1.06(\theta+d)}{1.06\theta+d}\right)^2$$

As $d \to \infty$, we have $\frac{\theta+d}{1.06\theta+d} \to 1$, so $r \to 1.06^2 = 1.1236$.

4.3 Semiparametric and Nonparametric methods

43 Using the kernel density distribution, the probability that a claim is larger than 3 is

$$\frac{1}{5}\left(\Phi\left(\frac{0.3-3}{0.8}\right)+\Phi\left(\frac{1.2-3}{0.8}\right)+\Phi\left(\frac{1.4-3}{0.8}\right)+\Phi\left(\frac{1.9-3}{0.8}\right)+\Phi\left(\frac{4.7-3}{0.8}\right)\right)=\frac{1}{5}\left(\Phi\left(-3.375\right)+\Phi\left(-2.25\right)+\Phi\left(-2\right)+\Phi\left(-1.37\right.\right.$$

44 (a)
For x in the interval $[2.7, 3.9]$, the survival function of the kernel smoothing density estimate is

$$S(x) = \frac{1}{5}\left(\frac{2.4+3-x}{6} + \frac{2.8+3-x}{6} + \frac{3.5+3-x}{6} + \frac{3.9+3-x}{6} + \frac{4.2+3-x}{6}\right) = 1.06 - \frac{x}{6}$$

Therefore, if it is in this interval, the median is the solution to

$$\begin{aligned} 1.06 - \frac{x}{6} &= 0.5 \\ x &= 6(1.06 - 0.5) \\ &= 3.18 \end{aligned}$$

(b)
Now the kernel density estimate is different in different intervals. By inspection, we see that the median will lie in the interval $[3.4, 3.7]$. In this interval, the survival function is

$$S(x) = \frac{1}{5}\left(0 + 0 + \frac{3.5+0.5-x}{1} + \frac{3.9+0.5-x}{1} + +1\right) = 1.88 - 0.4x$$

The median is therefore the solution to

$$\begin{aligned} 1.88 - 0.4x &= 0.5 \\ 0.4x &= 1.38 \\ x &= 3.45 \end{aligned}$$

45 The probability that the claim exceeds 3.5 is

$$\frac{1}{5}\left(\Phi\left(\frac{1.2-3.5}{2}\right)+\Phi\left(\frac{1.4-3.5}{2}\right)+\Phi\left(\frac{2.1-3.5}{2}\right)+\Phi\left(\frac{2.9-3.5}{2}\right)+\Phi\left(\frac{4.3-3.5}{2}\right)\right)=\frac{1}{5}\left(\Phi(-1.15)+\Phi(-1.05)+\Phi(-0.70)+\right.$$

46 For a Gamma distribution with parameters $\alpha = 3$ and $\theta = 1$, the probability of exceeding $\frac{5}{\theta}$ is

$$\int_{\frac{5}{\theta}}^{\infty} \frac{x^2 e^{-x}}{2}\,dx = \left[-\frac{x^2}{2}e^{-x}\right]_{\frac{5}{\theta}}^{\infty} + \int_{\frac{5}{\theta}}^{\infty} xe^{-x}\,dx = \frac{1}{2}\left(\frac{5}{\theta}\right)^2 e^{-\frac{5}{\theta}} + \frac{5}{\theta}e^{-\frac{5}{\theta}} + e^{-\frac{5}{\theta}} = e^{-\frac{5}{\theta}}\left(1 + \frac{5}{\theta} + \left(\frac{5}{\theta}\right)^2\right)$$

For the kernel density estimate, we calculate this for all observations

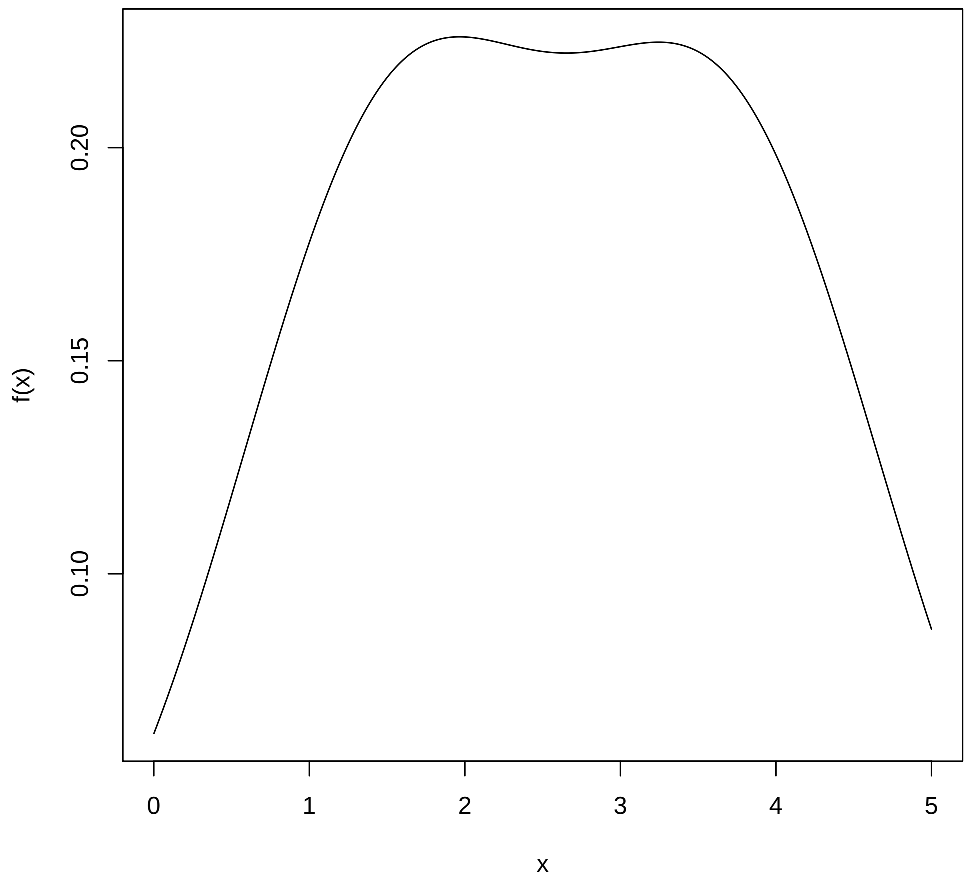
x	θ	$e^{-\frac{5}{\theta}}\left(1+\frac{5}{\theta}+\left(\frac{5}{\theta}\right)^2\right)$
1.8	0.6	0.01058961
, 2.1	0.7	0.02660242
, 2.1	0.7	0.02660242
2.4	0.8	0.05169997
3.6	1.2	0.21468531

The probability that the kernel density estimate exceeds 5 is therefore

$$\frac{0.01058961 + 0.02660242 + 0.02660242 + 0.05169997 + 0.21468531}{5} = 0.066035946$$

47

```
sigma<-seq_len(2000)/1000
Q47<-colMeans(pnorm((c(1.4, 1.9, 2.0, 2.8, 3.3)-3.1)%*%t(rep(1,length(sigma)))/(rep(1,5)%*%t(sigma))))
plot(sigma,Q47,type='l',xlab="Standard deviation",ylab="P(X 3.1)")
```



$$f(x) = \frac{1}{6\sqrt{2\pi}} \left(e^{-\frac{(x-1.4)^2}{2}} + e^{-\frac{(x-1.5)^2}{2}} + e^{-\frac{(x-1.7)^2}{2}} + e^{-\frac{(x-3.5)^2}{2}} + e^{-\frac{(x-3.7)^2}{2}} + e^{-\frac{(x-3.9)^2}{2}} \right)$$

49 The density function of X is $f_X(x) = x^{\alpha-1}(1-x)^{\beta-1}$ for $0 < x < 1$. The density function of $5X$ is therefore

$$f_{5X}(x) = \frac{1}{5} f_X\left(\frac{x}{5}\right) = \frac{1}{5}\left(\frac{x}{5}\right)^{\alpha-1}\left(1-\frac{x}{5}\right)^{\beta-1} = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{5^{\alpha+\beta-1}}$$

for $0 < x < 5$.

50 The density of X is $f_X(x) = \frac{x^{\alpha-x}e^{-\frac{x}{\theta}}}{\theta^\alpha \Gamma(\alpha)}$. The density of X^{-1} is therefore

$$f_{X^{-1}}(x) = x^{-2} f(x^{-1}) = x^{-2} \frac{x^{-(\alpha-1)} e^{-\frac{x^{-1}}{\theta}}}{\theta^\alpha \Gamma(\alpha)} = \frac{\theta^{-\alpha} e^{-\frac{\theta^{-1}}{x}}}{x^{\alpha+1} \Gamma(\alpha)}$$

51 The density of X is $f_X(x) = \frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}$. The density of X^2 is therefore

$$f_{X^2}(x) = \frac{f_X(\sqrt{x}) + f_X(-\sqrt{x})}{2\sqrt{x}} = \frac{2e^{-\frac{x}{2\sigma^2}}}{2\sqrt{\pi x}\sigma}$$

This is a Gamma distribution with shape $\alpha = \frac{1}{2}$ and scale $\theta = 2\sigma^2$.

52 The density of X is $f_X(x) = \frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}$. The density of e^X is therefore

$$f_{e^X}(x) = \frac{f_X(\log(x))}{x} = \frac{e^{-\frac{(\log(x)-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma x}$$

53 The survival function of X is $S_X(x) = \left(\frac{\theta}{\theta+x}\right)^{\alpha}$. Since $Y = \log\left(1+\frac{x}{\theta}\right)$ is an increasing function of X, we have $X = \theta(e^Y - 1)$, so

$$S_Y(x) = S_X\left(\theta\left(e^x - 1\right)\right) = \left(\frac{\theta}{\theta + \theta\left(e^x - 1\right)}\right)^{\alpha} = \left(\frac{1}{e^x}\right)^{\alpha} = e^{-\alpha x}$$

This is the survival function of an exponential distribution.

5.2.4 Mixture Distributions

54 The density of the inverse gamma distribution with $\alpha = 5$ is

$$f(x) = \frac{\theta^5 e^{-\frac{\theta}{x}}}{24x^6}$$

The density of the mixture distribution is therefore

$$\begin{aligned}
f(x) &= \int_0^\infty \frac{e^{-\frac{v}{7000}}}{7000} \frac{(1000+0.1v)^5 e^{-\frac{1000+0.1v}{x}}}{24x^6}\,dv \\
&= \int_0^\infty \frac{(1000+0.1v)^5 e^{-\left(\frac{1000+0.1v}{x}+\frac{v}{7000}\right)}}{168000x^6}\,dv \\
&= \int_{1000}^\infty \frac{u^5 e^{-\left(\frac{u}{x}+\frac{10u-10000}{7000}\right)}}{16800x^6}\,du \\
&= \frac{e^{\frac{10}{7}}}{16800x^6} \int_{1000}^\infty u^5 e^{-u\left(\frac{1}{x}+\frac{1}{700}\right)}\,du
\end{aligned}$$

Letting $t = \frac{700x}{x+700}$, we get

$$\begin{aligned}
\int_{1000}^\infty u^5 e^{-\frac{u}{t}}\,du &= \left[-u^5 t e^{-\frac{u}{t}}\right]_{1000}^\infty + \int_{1000}^\infty 5tu^4 e^{-\frac{u}{t}}\,du \\
&= \cdots \\
&= \left(1000^5 t + 5\times 1000^4 t + 20\times 1000^3 t^2 + 60\times 1000^2 t^3 + 120\times 1000 t^4 + 120t^5\right) e^{-\frac{1000}{t}}
\end{aligned}$$

Therefore, the density function of the amount paid on a random claim is

$$\begin{aligned}
f(x) = \frac{e^{\frac{10}{7}}}{16800x^6} &\left(1000^5\left(\frac{700x}{x+700}\right) + 5\times 1000^4\left(\frac{700x}{x+700}\right) + 20\times 1000^3\left(\frac{700x}{x+700}\right)^2\right. \\
&\left. +60\times 1000^2\left(\frac{700x}{x+700}\right)^3 + 120\times 1000\left(\frac{700x}{x+700}\right)^4 + 120\left(\frac{700x}{x+700}\right)^5\right) e^{-\frac{1000(x+700)}{700x}}
\end{aligned}$$

55 Given the hazard rate $\Theta = \theta$, the conditional probability that $X > 0.5$ is $e^{-0.5\theta}$. Therefore, the marginal probability is

$$\frac{1}{10}\int_1^{11} e^{-0.5\theta}\,d\theta = \frac{1}{10}\left[-2e^{-0.5\theta}\right]_1^{11} = \frac{e^{-0.5} - e^{-5.5}}{5} = 0.120488777655$$

56 The hazard rate is $\lambda(x) = \frac{f(x)}{S(x)} = \frac{-\frac{d}{dx}S(x)}{S(x)} = \frac{d}{dx}\log(S(x))$. Therefore the survival function is $S(x) = e^{-\int_0^x \lambda(t)\,dt}$. For this model, we have

$$\begin{aligned} S(75) &= e^{-\int_0^{10} 0.0001(20-x)\,dx - \int_{10}^{75} 10^{-5}x^2\,dx} \\ &= e^{-(0.002\times 10 - 0.0001\times\frac{10^2}{2}) - 10^{-5}\left(\frac{75^3-10^3}{3}\right)} \\ &= e^{-1.41791666667} \\ &= 0.242218112682 \end{aligned}$$

57 For a given risk factor Θ, the probability that a random claim exceeds \$1,000 is $\left(\frac{\Theta}{\Theta+1000}\right)^3$. The marginal probability that a random loss exceeds \$1,000 is therefore

$$\begin{aligned}
\mathbb{E}\left(\left(\frac{\Theta}{\Theta+1000}\right)^3\right) &= \int_0^\infty \frac{2\times 1000^2}{(1000+\theta)^3} \times \left(\frac{\theta}{\theta+1000}\right)^3 \, d\theta \\
&= 2000000 \int_{1000}^\infty \frac{(u-1000)^3}{u^6} \, du \\
&= 2000000 \int_{1000}^\infty (u^{-3} - 3000u^{-4} + 3000000u^{-5} - 10^9 u^{-6} \, du \\
&= 2000000 \left[-\frac{u^{-2}}{2} + 3000\frac{u^{-3}}{3} - 3000000\frac{u^{-4}}{4} + 10^9 \frac{u^{-5}}{5} \right]_{1000}^\infty \\
&= 2\left(\frac{1}{2} - 3\times\frac{1}{3} + 3\times\frac{1}{4} - \frac{1}{5}\right) \\
&= 0.1
\end{aligned}$$

58

(a) For a given value of θ, the probability that the time until a claim is at least 6 years is $1 - e^{-\frac{\theta^2}{6^2}}$. The probability that a random policyholder makes no claims for 6 years is therefore

$$\begin{aligned}\int_0^\infty \frac{1}{4}e^{-\frac{\theta}{4}}\left(1 - e^{-\frac{\theta^2}{6^2}}\right)d\theta &= 1 - \int_0^\infty \frac{1}{4}e^{-\frac{(\theta^2+9\theta)}{36}}\,d\theta \\ &= 1 - \int_0^\infty \frac{1}{4}e^{-\frac{(\theta^2+4.5)^2-4.5^2}{36}}\,d\theta \\ &= 1 - \frac{1}{4}e^{\frac{9}{16}}6\sqrt{\pi}\int_0^\infty \frac{e^{-\frac{(\theta^2+4.5)^2}{36}}}{6\sqrt{\pi}}\,d\theta \\ &= 1 - \frac{1}{4}e^{\frac{9}{16}}6\sqrt{\pi}\Phi\left(-\frac{4.5}{3\sqrt{2}}\right) \\ &= 0.3261073072\end{aligned}$$

(b) The probability that a policy makes no claim for 7 years is

$$\begin{aligned}\int_0^\infty \frac{1}{4}e^{-\frac{\theta}{4}}\left(1 - e^{-\frac{\theta^2}{7^2}}\right)d\theta &= 1 - \int_0^\infty \frac{1}{4}e^{-\frac{(\theta^2+12.25\theta)}{49}}\,d\theta \\ &= 1 - \int_0^\infty \frac{1}{4}e^{-\frac{(\theta^2+6.125)^2-6.125^2}{49}}\,d\theta \\ &= 1 - \frac{1}{4}e^{\frac{6.125^2}{49}}7\sqrt{\pi}\int_0^\infty \frac{e^{-\frac{(\theta^2+6.125)^2}{49}}}{7\sqrt{\pi}}\,d\theta \\ &= 1 - \frac{1}{4}e^{\frac{6.125^2}{49}}7\sqrt{\pi}\Phi\left(-\frac{6.125}{3.5\sqrt{2}}\right) \\ &= 0.279900503\end{aligned}$$

so the conditional probability of making no claim in the next year is $\frac{0.279900503}{0.3261073072} = 0.858307976608$.

59 For an individual with a given value of λ, the probability of surviving to 40 is

$$\begin{aligned} e^{-\int_0^{40} \lambda e^{0.08x}\,dx} &= e^{-\lambda\left[\frac{e^{0.08x}}{0.08}\right]_0^{40}} \\ &= e^{-\lambda\frac{e^{3.2}-1}{0.08}} \\ &= e^{-294.156627464\lambda} \end{aligned}$$

The marginal probability of an individual surviving to 40 is therefore

$$\begin{aligned} \int_0^\infty \frac{\lambda^2 e^{-100000\lambda}}{0.00001^3\Gamma(3)} e^{-294.156627464\lambda}\,d\lambda &= \frac{10^{15}}{2}\int_0^\infty \lambda^2 e^{-100294.156627464\lambda}\,d\lambda \\ &= \frac{10^{15}}{2} 100294.156627464^{-3}\Gamma(3) \\ &= 1.00294156627464^{-3} \\ &= 0.991226964653 \end{aligned}$$

For an individual with a given value of λ, the probability of surviving to 90 is

$$\begin{aligned} e^{-\int_0^{90} \lambda e^{0.08x}\,dx} &= e^{-\lambda\left[\frac{e^{0.08x}}{0.08}\right]_0^{90}} \\ &= e^{-\lambda\frac{e^{7.2}-1}{0.08}} \\ &= e^{-16730.3845549\lambda} \end{aligned}$$

The marginal probability of an individual surviving to 40 is therefore

$$\begin{aligned} \int_0^\infty \frac{\lambda^2 e^{-100000\lambda}}{0.00001^3\Gamma(3)} e^{-16730.3845549\lambda}\,d\lambda &= \frac{10^{15}}{2}\int_0^\infty \lambda^2 e^{-116730.3845549\lambda}\,d\lambda \\ &= \frac{10^{15}}{2} 116730.3845549^{-3}\Gamma(3) \\ &= 1.167303845549^{-3} \\ &= 0.628706935472 \end{aligned}$$

The probability of someone aged 40 surviving to 90 is therefore $\frac{0.628706935472}{0.991226964653} = 0.634271421069$.

60 The probability that a gamma distribution with $\alpha = 3$ and $\theta = 1000$ is less than 2000 is 0.3233236. Therefore the density at $2000 - \epsilon$ tends to

$$\frac{p}{0.3233236}\frac{2000^2 e^{-\frac{2000}{1000}}}{1000^3\Gamma(3)} = 0.000837150664145p$$

where p is the probability that a claim is small.

The probability that a Pareto distribution with $\alpha = 4$ and $\theta = 3000$ is more than 2000 is $\left(\frac{3000}{3000+2000}\right)^4 = 0.1296$. Therefore the density at $2000 + \epsilon$ tends to

$$\frac{q}{0.1296}\frac{4 \times 3000^4}{(3000+2000)^5} = 0.0008q$$

where $q = 1 - p$ is the probability that a claim is large. For the density to be continuous, these must be equal, i.e.

$$\begin{aligned}
0.000837150664145p &= 0.0008(1-p)\\
0.001637150664145p &= 0.0008\\
p &= 0.48865386523
\end{aligned}$$

61 Recall that for any random variable Y, $\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y|Z))$. Applying this to $Y = X$ and $Y = X^2$ gives

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\
&= \mathbb{E}(\mathbb{E}(X^2|Z)) - \mathbb{E}(\mathbb{E}(X|Z))^2 \\
&= \mathbb{E}(\mathbb{E}(X^2|Z)) - \mathbb{E}(\mathbb{E}(X|Z)^2) + \mathbb{E}(\mathbb{E}(X|Z)^2) - \mathbb{E}(\mathbb{E}(X|Z))^2 \\
&= \mathbb{E}(\mathbb{E}(X^2|Z) - \mathbb{E}(X|Z)^2) + \mathrm{Var}(\mathbb{E}(X|Z)) \\
&= \mathbb{E}(\mathrm{Var}((X|Z)) + \mathrm{Var}(\mathbb{E}(X|Z))
\end{aligned}$$

62 Let X be the cost of a random claim, and let Θ be the parameter for a random individual. The law of total variance gives that

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}(\mathrm{Var}(X|\Theta)) + \mathrm{Var}(\mathbb{E}(X|\Theta)) \\
&= \mathbb{E}\left(\frac{\alpha\Theta^2}{(\alpha-1)^2(\alpha-2)}\right) + \mathrm{Var}\left(\frac{\Theta}{\alpha-1}\right) \\
&= \frac{3}{2^2 \times 1}\mathbb{E}(\Theta^2) + \mathrm{Var}\left(\frac{\Theta}{2}\right) \\
&= \frac{3}{4}\left(\mathrm{Var}(\Theta) + \mathbb{E}(\Theta)^2\right) + \frac{1}{4}\mathrm{Var}(\Theta) \\
&= \frac{3}{4}\left(2 \times 3000^2 + (2 \times 3000)^2\right) + \frac{1}{4} \times 2 \times 3000^2 \\
&= 45000000
\end{aligned}$$

63 For a given policy with parameter θ, the conditional survival function is the conditional distribution function of a gamma distribution. That is,

$$\begin{aligned}
S_\theta(x) &= \int_0^{\frac{\theta}{x}} \frac{z^{\alpha-1}e^{-z}}{\Gamma(\alpha)}\,dz \\
&= \int_0^{\frac{\theta}{x}} ze^{-z}\,dz \\
&= \left[-ze^{-z}\right]_0^{\frac{\theta}{x}} + \int_0^{\frac{\theta}{x}} e^{-z}\,dz \\
&= 1-\left(1+\frac{\theta}{x}\right)e^{-\frac{\theta}{x}}
\end{aligned}$$

The survival function of a random policy is

$$\begin{aligned}
S(x) &= \mathbb{E}(S_\Theta(x)) \\
&= \mathbb{E}\left(1-\left(1+\frac{\Theta}{x}\right)e^{-\frac{\Theta}{x}}\right) \\
&= \int_0^\infty \frac{\theta e^{-\frac{\theta}{1000}}}{1000^2\Gamma(2)}\left(1-\left(1+\frac{\theta}{x}\right)e^{-\frac{\theta}{x}}\right)d\theta \\
&= 1-\int_0^\infty \frac{e^{-\theta\left(\frac{1}{1000}+\frac{1}{x}\right)}}{1000^2}\left(\theta+\frac{\theta^2}{x}\right)d\theta \\
&= 1-\frac{x^2}{(x+1000)^2}\int_0^\infty \frac{\theta e^{-\frac{\theta}{\left(\frac{1000x}{x+1000}\right)}}}{\left(\frac{1000x}{x+1000}\right)^2}\,d\theta - \frac{2000x^3}{x(x+1000)^3}\int_0^\infty \frac{\theta^2 e^{-\frac{\theta}{\left(\frac{1000x}{x+1000}\right)}}}{\left(\frac{1000x}{x+1000}\right)^3\Gamma(3)}\,d\theta \\
&= 1-\frac{x^2}{(x+1000)^2}-\frac{2000x^2}{(x+1000)^3}
\end{aligned}$$

The VaR is therefore the solution to

$$\begin{aligned}
1-\frac{x^2}{(x+1000)^2}-\frac{2000x^2}{(x+1000)^3} &= 0.05 \\
\frac{x^2}{(x+1000)^2}+\frac{2000x^2}{(x+1000)^3} &= 0.95 \\
x^2(x+1000)+2000x^2 &= 0.95(x+1000)^3
\end{aligned}$$

Numerically, we get $x = 6388.232908696$

Now the TVaR is

$$\frac{1}{0.05}\int_{\mathrm{VaR}_{0.95}}^\infty S(x)\,dx = 20\int_{6388.232908696}^\infty \left(1-\frac{x^2}{(x+1000)^2}-\frac{2000x^2}{(x+1000)^3}\right)dx$$

This integral is undefined.

64 Let $Z = 1$ if X is in the component with mean μ_1 and variance ${\sigma_1}^2$, $Z = 2$ if X is in the component with mean μ_2 and variance ${\sigma_2}^2$, and $Z = 3$ if X is in the component with mean μ_3 and variance ${\sigma_3}^2$.

Now we have

$$\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|Z)) = p_1\mu_1 + p_2\mu_2 + p_3\mu_3$$

and

$$\begin{aligned}\mathrm{Var}(X) &= \mathbb{E}(\mathrm{Var}(X|Z)) + \mathrm{Var}(\mathbb{E}(X|Z)) \\ &= p_1{\sigma_1}^2 + p_2{\sigma_2}^2 + p_3{\sigma_3}^2 + p_1{\mu_1}^2 + p_2{\mu_2}^2 + p_3{\mu_3}^2 - (p_1\mu_1 + p_2\mu_2 + p_3\mu_3)^2\end{aligned}$$

65

For thefts, the proportion of claims above 10000 is

$$1 - \left(1 + \frac{4000}{10000}\right)^{-3} = 0.63556851311953352770$$

For collisions, the proportion of claims above 10000 is

$$e^{-100}\left(1 + 100 + \cdots + \frac{100^{50}}{50!}\right) = 2.401592e - 08$$

For other claims, the proportions above 10000 is $e^{-10000/3000} = 0.03567399$

The total proportion is therefore $0.15 \times 0.63556851311953352770 + 0.75 \times 2.401592e - 08 + 0.1 \times 0.03567399 = 0.09890269$

66 The density function of the transformed beta distribution is

$$f_X(x) = \left(\frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)}\right) \frac{\gamma\left(\frac{x}{\theta}\right)^{\gamma\tau}}{x\left(1+\left(\frac{x}{\theta}\right)^{\gamma}\right)^{\alpha+\tau}}$$

so the density of $\frac{1}{X}$ is

$$\begin{aligned}
f_{\frac{1}{X}}(x) &= \frac{1}{x^2} f_X\left(\frac{1}{x}\right) \\
&= \frac{1}{x^2}\left(\frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)}\right) \frac{\gamma\left(\frac{1}{\theta x}\right)^{\gamma\tau}}{x^{-1}\left(1+\left(\frac{1}{\theta x}\right)^{\gamma}\right)^{\alpha+\tau}} \\
&= \left(\frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)}\right) \frac{\gamma\left(\frac{1}{\theta x}\right)^{\gamma\tau}(\theta x)^{\gamma(\alpha+\tau)}}{x\left(1+\left(\frac{1}{\theta x}\right)^{\gamma}\right)^{\alpha+\tau}(\theta x)^{\gamma(\alpha+\tau)}} \\
&= \left(\frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)}\right) \frac{\gamma(\theta x)^{\gamma\alpha}}{x\left((\theta x)^{\gamma}+1\right)^{\alpha+\tau}}
\end{aligned}$$

If we set $\phi = \frac{1}{\theta}$ then

$$f_{\frac{1}{X}}(x) = \left(\frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)}\right) \frac{\gamma\left(\frac{x}{\phi}\right)^{\gamma\alpha}}{x\left(1+\left(\frac{x}{\phi}\right)^{\gamma}\right)^{\alpha+\tau}}$$

This is a transformed beta distribution with $\alpha = \tau$, $\tau = \alpha$ and $\theta = \frac{1}{\theta}$.

67 The density function of the transformed beta distribution is

$$f_X(x) = \left(\frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)}\right) \frac{\gamma \left(\frac{x}{\theta}\right)^{\gamma\tau}}{x\left(1+\left(\frac{x}{\theta}\right)^{\gamma}\right)^{\alpha+\tau}}$$

We rearrange this as

$$f_X(x) = \left(\frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)}\right) \left(\frac{x^\gamma}{\theta^\gamma + x^\gamma}\right)^{\tau} \frac{\gamma}{x\left(1+\left(\frac{x}{\theta}\right)^{\gamma}\right)^{\alpha}}$$

If $\theta\tau^{\frac{1}{\gamma}} = \xi$, then $\theta^\gamma = \frac{\xi^\gamma}{\tau}$. Substituting this into the density gives

$$f_X(x) = \left(\frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)}\right) \left(\frac{\tau x^\gamma}{\xi^\gamma + \tau x^\gamma}\right)^{\tau} \frac{\gamma}{x\left(1+\tau\left(\frac{x}{\xi}\right)^{\gamma}\right)^{\alpha}}$$

As $\tau \to \infty$, we have $\frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)} \to \tau^\alpha$. Substituting this in gives

$$\begin{aligned}
f_X(x) &\to \frac{1}{\Gamma(\alpha)} \left(\frac{\tau x^\gamma}{\xi^\gamma + \tau x^\gamma}\right)^{\tau} \frac{\gamma\tau^\alpha}{x\left(1+\tau\left(\frac{x}{\xi}\right)^{\gamma}\right)^{\alpha}} \\
&= \frac{\gamma}{x\Gamma(\alpha)} \left(1+\left(\frac{\xi}{x}\right)^{\gamma}\tau^{-1}\right)^{\tau} \left(\tau^{-1}+\left(\frac{x}{\xi}\right)^{\gamma}\right)^{-\alpha} \\
&= \frac{\gamma}{x\Gamma(\alpha)} \left(\frac{\xi}{x}\right)^{\gamma\alpha} \left(1+\left(\frac{\xi}{x}\right)^{\gamma}\tau^{-1}\right)^{\tau} \left(1+\left(\frac{\xi}{x}\right)^{\gamma}\tau^{-1}\right)^{-\alpha} \\
&\to \frac{\gamma}{x\Gamma(\alpha)} \left(\frac{\xi}{x}\right)^{\gamma\alpha} e^{-\left(\frac{\xi}{x}\right)^{\gamma}}
\end{aligned}$$

This is the density of an inverse transformed gamma distribution with $\tau = \gamma$ and $\theta = \xi$.

68

The density of the transformed gamma distribution is

$$f(x)=\frac{\tau x^{\alpha\tau}e^{-\left(\frac{x}{\theta}\right)^{\tau}}}{x\Gamma(\alpha)\theta^{\alpha\tau}}$$

We substitute Stirlings formula:

$$\Gamma(\alpha)\approx e^{-\alpha}\alpha^{\alpha-\frac{1}{2}}\sqrt{2\pi}$$

Now the conditions given imply $\theta^{\tau}=\sigma^2\tau^2$ and $\alpha=\frac{\mu}{\sigma^2\tau}+\frac{1}{\sigma^2\tau^2}$. We see that $(\alpha\theta^{\tau})^{\alpha}=(1+\mu\tau)^{\alpha}$. Since $\alpha=\frac{\mu}{\sigma^2\tau}+\frac{1}{\sigma^2\tau^2}=\frac{\mu}{\sigma^2\tau}\left(1+\frac{1}{\mu\tau}\right)$, we have that

$$(1+\mu\tau)^{\alpha}=(1+\mu\tau)^{\frac{\mu}{\sigma^2\tau}}\left((1+\mu\tau)^{\frac{1}{\mu\tau}}\right)^{\frac{\mu}{\sigma^2\tau}}=e^{\frac{\mu^2}{\sigma^2}}e^{\frac{\mu}{\sigma^2\tau}}$$

Substituting this into the formula, we have that

$$f(x)=\frac{\tau\sqrt{\alpha}x^{\alpha\tau}e^{-\left(\frac{x}{\theta}\right)^{\tau}}e^{\alpha}}{xe^{\frac{\mu}{\sigma^2\tau}+\frac{\mu^2}{\sigma^2}}\sqrt{2\pi}}$$

We have that $\theta^{\tau}=\sigma^2\tau^2$, and $x^{\tau}=e^{\tau\log x}=1+\tau\log(x)+\frac{\tau^2(\log(x))^2}{2}+\ldots$. Therefore $\left(\frac{x}{\theta}\right)^{\tau}=\frac{1}{\sigma^2\tau^2}+\frac{\log(x)}{\sigma^2\tau}+\frac{(\log(x))^2}{2\sigma^2}+\ldots$. Substituting this in we get

$$f(x)=\frac{\tau\sqrt{\alpha}x^{\alpha\tau}e^{\alpha-\frac{1}{\sigma^2\tau^2}-\frac{\log(x)}{\sigma^2\tau}-\frac{(\log(x))^2}{2\sigma^2}-\frac{\mu}{\sigma^2\tau}-\frac{\mu^2}{\sigma^2}}}{x\sqrt{2\pi\alpha}}$$

recalling $\alpha=\frac{1}{\sigma^2\tau^2}+\frac{\mu}{\sigma^2\tau}$, we get

$$f(x)=\frac{\tau\sqrt{\alpha}x^{\alpha\tau}e^{-\frac{\log(x)}{\sigma^2\tau}-\frac{(\log(x))^2}{2\sigma^2}-\frac{\mu^2}{\sigma^2}}}{x\sqrt{2\pi}}=\frac{\tau\sqrt{\alpha}e^{\log(x)\left(\alpha\tau-\frac{1}{\sigma^2\tau}\right)-\frac{(\log(x))^2}{2\sigma^2}-\frac{\mu^2}{\sigma^2}}}{x\sqrt{2\pi}}$$

Since $\alpha\tau-\frac{1}{\sigma^2\tau}=\frac{\mu}{\sigma^2}$, we have obtained

$$f(x)=\frac{\tau\sqrt{\alpha}e^{\frac{2\mu\log(x)-(\log(x))^2}{2\sigma^2}-\frac{\mu^2}{\sigma^2}}}{x\sqrt{2\pi}}$$

Finally, as $\tau\sqrt{\alpha}=\sqrt{\frac{1}{\sigma^2}+\frac{\mu\tau}{\sigma^2}}$, as $\tau\to 0$, this becomes $\frac{1}{\sigma}$, so

$$f(x)=\frac{e^{\frac{2\mu\log(x)-(\log(x))^2}{2\sigma^2}-\frac{\mu^2}{\sigma^2}}}{x\sqrt{2\pi}\sigma}=\frac{e^{\frac{(\log(x)-\mu)^2}{\sigma^2}}}{x\sqrt{2\pi}\sigma}$$

which is the density of a log-normal distribution.

69

Let $[a, b]$ be the support of the distribution (a and b can be $\pm\infty$). We know that $\int_a^b p(x)e^{r(\theta)x}dx = q(\theta)$. Differentiating with respect to θ gives

$$q'(\theta) = \frac{d}{d\theta}\int_a^b p(x)e^{r(\theta)x}dx = \int_a^b p(x)\frac{d}{d\theta}e^{r(\theta)x}dx = \int_a^b p(x)xr'(\theta)e^{r(\theta)x}dx = r'(\theta)\mu(\theta)q(\theta)$$

This gives $\mu(\theta) = \frac{q'(\theta)}{r'(\theta)q(\theta)}$.

Differentiating again gives

$$\mu'(\theta)q(\theta) + \mu(\theta)q'(\theta) = \frac{d}{d\theta}\int_a^b xp(x)e^{r(\theta)x}dx = \int_a^b r'(\theta)x^2p(x)e^{r(\theta)x}dx = r'(\theta)q(\theta)\mu_2'(\theta)$$

Dividing through by $r'(\theta)q(\theta)$, we get

$$\frac{\mu'(\theta)}{r'(\theta)} + \mu(\theta)^2 = \mu_2'(\theta)$$

Since the variance is $\mu_2'(\theta) - \mu(\theta)^2$, this is equal to $\frac{\mu'(\theta)}{r'(\theta)}$.

70 The Gamma distribution is linear exponential with $p(x) = x^{\alpha-1}$, $r(\theta) = -\theta^{-1}$ and $q(\theta) = \theta^{\alpha}\Gamma(\alpha)$. The mean is therefore

$$\mu(\theta) = \frac{q'(\theta)}{r'(\theta)q(\theta)} = \frac{\alpha\theta^{\alpha-1}\Gamma(\alpha)}{\theta^{-2}\theta^{\alpha}\Gamma(\alpha)} = \alpha\theta$$

The variance is

$$\frac{\mu'(\theta)}{r'(\theta)} = \frac{\alpha}{\theta^{-2}} = \alpha\theta^2$$

71 (a) The PGF of the Poisson distribution is given by

$$\begin{aligned}
P(z) &= \mathbb{E}(z^X) \\
&= \sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} z^n \\
&= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda z)^n}{n!} \\
&= e^{-\lambda} e^{\lambda z} \\
&= e^{-\lambda(1-z)}
\end{aligned}$$

(b) If X_1 and X_2 are independent Poisson random variables with parameters λ_1 and λ_2 respectively, then

$$P_{X_1+X_2}(z) = P_{X_1}(z)P_{X_2}(z) = e^{-\lambda_1(1-z)}e^{-\lambda_2(1-z)} = e^{-(\lambda_1+\lambda_2)(1-z)}$$

This is the PGF of a Poisson distribution with mean $\lambda_1 + \lambda_2$.

72 Let N be the number of losses and let C be the number of claims. We have that $N \sim \text{Poisson}(\lambda)$ and $C|N \sim B(N,p)$. Therefore we have $P(C=m|N=n) = \binom{n}{m}p^m(1-q)^{n-m}$. Therefore

$$\begin{aligned}
P(C=m) &= \sum_{n=0}^{\infty} P(N=n)P(C=m|N=n) \\
&= \sum_{n=m}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!}\binom{n}{m}p^m(1-p)^{n-m} \\
&= e^{-\lambda}\frac{p^m}{m!}\sum_{n=m}^{\infty}\frac{\lambda^n}{(n-m)!}(1-p)^{n-m} \\
&= e^{-\lambda}\frac{p^m\lambda^m}{m!}\sum_{n=m}^{\infty}\frac{(\lambda(1-p))^{n-m}}{(n-m)!} \\
&= e^{-\lambda}\frac{p^m\lambda^m}{m!}e^{\lambda(1-p)} \\
&= e^{-\lambda p}\frac{(\lambda p)^m}{m!}
\end{aligned}$$

This is the probability mass function for a Poisson distribution with mean λp.

We can also show this more easily with probability generating functions. Let $Z_1, \dots, Z_N$ be indicator variables of whether a loss leads to a claim. Then $C = Z_1 + \cdots + Z_N$, so

$$\mathbb{E}(z^C|N) = \mathbb{E}(z^{Z_1}\cdots z^{Z_n}|N) = \mathbb{E}(z^{Z_1})^N = ((1-p)+pZ)^N$$

Therefore

$$P_C(z) = \mathbb{E}(z^C) = \mathbb{E}(\mathbb{E}(z^C|N)) = \mathbb{E}(((1-p)+pz)^N) = p_N((1-p)+pz) - e^{-\lambda(1-(1-p+pz))} = e^{-\lambda p(1-z)}$$

which is the PGF of a Poisson distribution with mean λp.

Next we need to show that the C and $N-C$ are independent. We have that

$$\begin{aligned}
P(C=m, N-C=k) &= P(C=m, N=m+k) \\
&= P(N=m+k)P(C=m|N=m+k) \\
&= e^{-\lambda}\frac{\lambda^{m+k}}{(m+k)!}\binom{m+k}{m}p^m(1-p)^k \\
&= e^{-\lambda}\frac{(\lambda p)^m(\lambda(1-p))^k}{m!k!}
\end{aligned}$$

As this separates as a product of functions of m and k, we see that C and $N-C$ are independant.

73 (a) The probability that the number of claims is zero is $0.8^{10} = 0.1073741824$.
(b) The probability that the number of claims is three is $\binom{10}{3}0.2^3 0.8^7 = 0.201326592$.

74 A binomial random variable is a sum of independant Bernoulli random variables. Let the Bernoulli random variables be $Z_1, \dots, Z_n$, where each $P(Z_i = 1) = p$. This gives $P_{Z_i}(z) = \mathbb{E}(z^{Z_i}) = (1-p) + pz$. We have $X = Z_1 + \cdots + Z_n$, so $P_X(z) = P_{Z_1}(z) \cdots P_{Z_n}(z) = P_{Z_1}(z)^n = (1-p+pz)^n$.

75

If X is a gamma mixture of Poisson random variables with parameters α and θ, then

$$P(X=n)=\int_0^\infty e^{-\lambda}\frac{\lambda^n}{n!}\frac{\lambda^{\alpha-1}}{\theta^\alpha\Gamma(\alpha)}e^{-\frac{\lambda}{\theta}}d\lambda=\frac{1}{n!\Gamma(\alpha)\theta^\alpha}\int_0^\infty\lambda^{n+\alpha-1}e^{-\lambda\left(1+\frac{1}{\theta}\right)}d\lambda$$

We make the substitution $\tau=\lambda\left(1+\frac{1}{\theta}\right)$, then we have $d\tau=\left(1+\frac{1}{\theta}\right)d\lambda$ the integral becomes

$$\left(1+\frac{1}{\theta}\right)^{-(n+\alpha)}\int_0^\infty\tau^{n+\alpha-1}e^{-\tau}d\tau=\Gamma(n+\alpha)\left(1+\frac{1}{\theta}\right)^{n+\alpha}$$

and so

$$P(X=n)=\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}\left(\frac{\theta}{1+\theta}\right)^n\left(\frac{1}{1+\theta}\right)^\alpha$$

A negative binomial distribution with $r=\alpha$ and $\beta=\theta$

76

X has a negative binomial distribution with $r = 70$, $\beta = 0.08$, so $P(X = 0) = \left(\frac{1}{1.08}\right)^{70} = 0.004574431$, so prob at least one claim is $1 - 0.004574431 = 0.99543$.

77 Let X be a negative binomial random variable with parameters r and β.

$$\begin{aligned}
P_X(z) &= \sum_{n=0}^{\infty} \binom{n+r-1}{n} \left(\frac{\beta}{1+\beta}\right)^n \left(\frac{1}{1+\beta}\right)^r z^n \\
&= \left(\frac{1}{1+\beta}\right)^r \sum_{n=0}^{\infty} \binom{n+r-1}{n} \left(\frac{\beta z}{1+\beta}\right)^n \\
&= \left(\frac{1}{1+\beta}\right)^r \left(1 - \frac{\beta z}{1+\beta}\right)^{-r} \\
&= (1+\beta-\beta z)^{-r}
\end{aligned}$$

78 The PGF of the negative binomial distribution is $P(z) = (1 + \beta - \beta z)^{-r}$. If $r\beta = \lambda$, then this becomes

$$P(z) = \left(1 + \frac{\lambda(1-z)}{r}\right)^{-r} \to e^{-\lambda(1-z)}$$

as $r \to \infty$. Therefore, we have that the limit of the negative binomial distribution is Poisson with mean λ.

79

Binomial

$$\frac{p_k}{p_k - 1} = \frac{\binom{n}{k} p^k (1-p)^{n-k}}{\binom{n}{k-1} p^{k-1} (1-p)^{n-k+1}} = \frac{(n-k)p}{k(1-p)} = -\frac{p}{1-p} + \frac{\frac{np}{1-p}}{k}$$

This is from the $(a, b, 0)$ class with $a = -\frac{p}{1-p}$ and $b = \frac{np}{1-p}$.

Poisson

$$\frac{p_k}{p_k - 1} = \frac{e^{-\lambda} \frac{\lambda^k}{k!}}{e^{-\lambda} \frac{\lambda^{k-1}}{(k-1)!}} = \frac{\lambda}{k}$$

This is from the $(a, b, 0)$ class with $a = 0$ and $b = \lambda$.

Negative Binomial

$$\frac{p_k}{p_k - 1} = \frac{\binom{k+r-1}{k} \left(\frac{\beta}{1+\beta}\right)^k \left(\frac{1}{1+\beta}\right)^r}{\binom{k+r-2}{k-1} \left(\frac{\beta}{1+\beta}\right)^{k-1} \left(\frac{1}{1+\beta}\right)^r} = \frac{(k+r-1)\beta}{k(1+\beta)} = \frac{\beta}{1+\beta} + \frac{(r-1)\frac{\beta}{1+\beta}}{k}$$

This is from the $(a, b, 0)$ class with $a = \frac{\beta}{1+\beta}$ and $b = \frac{(r-1)\beta}{1+\beta}$.

Conversely, suppose X is from the $(a, b, 0)$ class. We ignore the degenerate case $P(X = 0) = 1$, so $p_0 > 0$ and $p_1 > 0$. If $a > 0$, we must have $b > -a$, since otherwise $\frac{p_1}{p_0} \leqslant 0$. If $a > 1$, then for all sufficiently large k, we have $p_k > p_{k+1}$, which is impossible for a distribution. Therefore, we must have $a \leqslant 1$.

If $a = 1$, then $b > -1$, we have $p_n = \left(1 - \frac{b}{n}\right) p_{n-1} > \left(1 - \frac{1}{n}\right) p_{n-1} = \frac{n-1}{n} p_{n-1}$. Iterating this give $p_n > \frac{1}{n} p_0$, and again $\sum_{n=0}^{\infty} p_n$ does not converge, so we do not get a distribution.

If $0 < a < 1$, then let $\beta = \frac{a}{1-a}$, and $r = \frac{b}{a} + 1$, and we see that X follows a negative binomial distribution with parameters r and β.

If $a = 0$, then X follows a Poisson distribution with mean $\lambda = b$.

If $a < 0$, then for large enough n, we have $a + \frac{b}{n} < 0$, so $p_n = \left(a + \frac{b}{n}\right) p_{n-1}$ is only possible if $p_n = p_{n-1} = 0$. To achieve this, there must be some N such that $a + \frac{b}{N} = 0$. That is, we must have $b = -Na$ for some N. If we let $p = \frac{-a}{-a+1}$, then we see that $a = -\frac{p}{1-p}$ and $b = \frac{Np}{1-p}$, so X follows a binomial distribution with parameters N and p.

80

k	$\frac{kp_k}{p_{k-1}}$
1	122/861=0.141695703
2	26/122=0.213114754
3	15/13=1.153846154
4	4/3=1.3333333
5	0

This ratio is increasing, so we have $b < 0$. This suggests a negative binomial distribution is most appropriate.

6.6 Truncation and Modification at zero

81

$$
\begin{aligned}
P(X = 0) &= 1.06^{-3.4} = 0.8202761 \\
P(X = 1) &= 3.4 \times 1.06^{-3.4}\frac{0.06}{1.06} = 0.157864457 \\
P(X = 2) &= 4.4 \times 3.4 \times 1.06^{-3.4}\frac{0.06^2}{1.06^2} = 0.019658593
\end{aligned}
$$

So $P(X > 0) = 0.1797$, $P(X > 3) = 0.00220085$, so the probability that the zero-truncated random variable is at least 3 is $\frac{0.00220085}{0.1797} = 0.012245728$.

82 The distribution is a zero-modified negative binomial distribution. Without the modification, the probability that an individual makes no claims is $1.7^{-0.8} = 0.654095050552$. Therefore the probability that a random individual buys the policy is $1 - \frac{0.654095050552}{2} = 0.672952474724$. The probability that an insured individual makes 0 claims is therefore $\frac{0.654095050552}{2 \times 0.672952474724} = 0.485989037205$. The probability that an individual makes n claims, for $n > 0$ is

$$\frac{1}{0.672952474724}(1.7)^{-0.8}\binom{n+r-1}{n}\left(\frac{7}{17}\right)^n$$

83

(a) For truncated $(a, b, 1)$ distribution, since the probabilities sum to 1 we have

$$1 = p_1\left(1 + \left(a + \frac{b}{2}\right) + \left(a + \frac{b}{2}\right)\left(a + \frac{b}{3}\right) + \ldots\right)$$

Now we have $a = \frac{\beta}{1+\beta}$ and $b = \frac{(r-1)\beta}{1+\beta}$, so $r = \frac{b}{a} + 1$, and

$$1 = p_1\left(1 + \left(\frac{1+r}{2}a\right) + \left(\frac{1+r}{2}a\right)\left(\frac{2+r}{3}a\right) + \ldots\right)$$

Multiplying out each term and multiplying by ra gives

$$1 = \frac{p_1}{ra}\left(ra + \frac{r(1+r)}{2}a^2 + \frac{r(1+r)(2+r)}{3!}a^3 + \ldots\right)$$

The series in the brackets is a binomial expansion of $(1-a)^{-r} - 1$, so we have

$$p_1 = \frac{ra}{(1-a)^{-r} - 1} = \frac{r\beta}{(1+\beta)((1+\beta)^r - 1)}$$

(b) The expected value is given by

$$\begin{aligned}
\mathbb{E}(X) &= p_1 + 2\left(a + \frac{b}{2}\right)p_1 + 3\left(a + \frac{b}{2}\right)\left(a + \frac{b}{3}\right)p_1 + \ldots \\
&= p_1\left(1 + 2a + 3a\left(a + \frac{b}{2}\right) + 4a\left(a + \frac{b}{2}\right)\left(a + \frac{b}{3}\right) + \ldots + b + b\left(a + \frac{b}{2}\right) + b\left(a + \frac{b}{2}\right)\left(a + \frac{b}{3}\right) + \ldots\right) \\
&= p_1 + a + a\mathbb{E}(X) + b
\end{aligned}$$

We solve this to get

$$\mathbb{E}(X) = \frac{p_1 + a + b}{1 - a}$$

For the ETNB, this is

$$\mathbb{E}(X) = \frac{p_1 + \frac{(r-1)\beta}{1+\beta} + \frac{\beta}{1+\beta}}{1 - \frac{\beta}{1+\beta}} = p_1(1+\beta) + r\beta$$

Substituting p_1 from (a) gives

$$\mathbb{E}(X) = \frac{r\beta}{((1+\beta)^r - 1)} + r\beta = r\beta\left(1 + \frac{1}{((1+\beta)^r - 1)}\right) = \frac{r\beta}{(1 - (1+\beta)^{-r})}$$

(c) If $r = 0$, then $b = -a$ and we have $\frac{p_n}{p_{n-1}} = a - \frac{a}{n} = a\frac{n-1}{n}$, so $p_n = \frac{p_1 a^{n-1}}{n}$. We therefore get

$$1 = p_1\left(1 + \frac{a}{2} + \frac{a^2}{3} + \cdots\right) = \frac{-p_1\log(1-a)}{a}$$

Since $a = \frac{\beta}{1+\beta}$, we have $-\log(1-a) = \log(1+\beta)$ This gives

$$p_1 = \frac{\beta}{(1+\beta)\log(1+\beta)}$$

(d) The expectation of a logarithmic distribution is

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{n=1}^{\infty} n p_n \\
&= \sum_{n=1}^{\infty} n p_1 \frac{a^{n-1}}{n} \\
&= p_1 \sum n = 1^{\infty} a^{n-1} \\
&= \frac{p_1}{1-a} \\
&= \frac{\beta}{\log(1+\beta)}
\end{aligned}$$

84 (a) From Question 83(a), we have

$$p_1 = \frac{r\beta}{(1+\beta)((1+\beta)^r - 1)} = \frac{-0.6 \times 0.8}{1.8(1.8^{-0.6} - 1)} = 0.897286525399$$

We also have $a = \frac{\beta}{1+\beta} = \frac{0.8}{1.8} = \frac{4}{9}$ and $b = -1.6a$, so $\frac{p_n}{p_{n-1}} = \frac{4}{9}\left(1 - \frac{1.6}{n}\right)$. This gives

$$p_n = 0.897286525399 \left(\frac{4}{9}\right)^{n-1} \left(1 - \frac{1.6}{2}\right) \cdot \left(1 - \frac{1.6}{n}\right)$$

From Question 83(b),

$$\mathbb{E}(X) = \frac{p_1 + a + b}{1 - a} = \frac{0.897286525399 - 0.6 \times \frac{4}{9}}{\frac{5}{9}} = 1.13511574572$$

(b) From Question 83(c), we have

$$p_1 = \frac{\beta}{(1+\beta)\log(1+\beta)} = \frac{0.5}{1.5\log(1.5)} = 0.822101154126$$

We also have $a = \frac{\beta}{1+\beta} = \frac{0.5}{1.5} = \frac{1}{3}$ and $b = -a$, so $\frac{p_n}{p_{n-1}} = \frac{1}{3}\left(\frac{n-1}{n}\right)$ and $p_n = \frac{p_1}{3^{n-1}n} = \frac{2.46630346238}{3^n n}$.

From Question 83(d), we have

$$\mathbb{E}(X) = \frac{\beta}{\log(1+\beta)} = \frac{0.5}{\log(1.5)} = 1.23315173119$$

8 Frequency and Severity with Coverage Modifications

8.2 Deductibles

85 The Burr distribution has survival function

$$S(x) = \frac{1}{\left(1 + \left(\frac{x}{\theta}\right)^{\gamma}\right)^{\alpha}}$$

(a)
The new distribution has survival function

$$S(x) = \frac{\left(\frac{1}{\left(1+\left(\frac{x+1000}{\theta}\right)^{\gamma}\right)^{\alpha}}\right)}{\left(\frac{1}{\left(1+\left(\frac{1000}{\theta}\right)^{\gamma}\right)^{\alpha}}\right)} = \left(\frac{\theta^{\gamma} + 1000^{\gamma}}{\theta^{\gamma} + (x+1000)^{\gamma}}\right)^{\alpha}$$

(b) with a franchise deductible, expected value of the claim is

$$1000 + \int_{1000}^{\infty} \left(\frac{\theta^{\gamma} + 1000^{\gamma}}{\theta^{\gamma} + x^{\gamma}}\right)^{\alpha} dx$$

Numerically, we see this integral is 33442.51

8.3 Loss Elimination Ratio and the Effect of Inflation

86

The inverse Gamma has pdf

$$f(x) = \frac{\left(\frac{\theta}{x}\right)^{\alpha} e^{-\frac{\theta}{x}}}{x\Gamma(\alpha)} = \frac{\left(\frac{2000}{x}\right)^{3.2} e^{-\frac{2000}{x}}}{2.423965x}$$

The expected value is $\frac{2000}{2.2}$. On the other hand, if the company introduces a deductible of \$500, the expected claim value per loss is

$$\int_{500}^{\infty} (x-500)\frac{\left(\frac{2000}{x}\right)^{3.2} e^{-\frac{2000}{x}}}{2.423965x} dx = \int_{500}^{\infty} x\frac{\left(\frac{2000}{x}\right)^{3.2} e^{-\frac{2000}{x}}}{2.423965x} dx - 500\int_{500}^{\infty} \frac{\left(\frac{2000}{x}\right)^{3.2} e^{-\frac{2000}{x}}}{2.423965x} dx$$

We substitute $y = \frac{2000}{x}$ with $\frac{dy}{dx} = -\frac{2000}{x^2} = -\frac{y^2}{2000}$. Now the expected claim per loss is

$$\int_{0}^{4} \frac{2000y^{1.2}e^{-y}}{2.423965} dx - 500\int_{0}^{4} \frac{2000y^{0.2}e^{-y}}{2.423965} dx = 553.0085 - 276.2932 = 276.7153$$

So the loss elimination ratio is $1 - \frac{276.7153\times 2.2}{2000} = 1 - 0.30438683 = 0.69561317$.

87

wlog $\theta = 1$, so the expected loss without a deductible is $\frac{1}{\alpha} = \frac{1}{2}$. With a deductible, the expected loss is

$$\int_2^\infty \frac{1}{(1+x)^2} dx = \int_3^\infty y^{-2} dy = \frac{1}{18}$$

The loss elimination ratio is therefore $\frac{8}{9}$.

After inflation of 100%, θ has doubled, so the deductible is now equal to θ. We can again assume wlog that $\theta = 1$, so with the deductible, the expected loss is

$$\int_1^\infty \frac{1}{(1+x)^2} dx = \int_2^\infty y^{-2} dy = \frac{1}{8}$$

The loss elimination ratio is therefore $\frac{3}{4}$.

88
(a) The Weibull distribution has survival function $e^{-\left(\frac{x}{\theta}\right)^{\tau}}$, so the expected loss is

$$\int_0^\infty e^{-\left(\frac{x}{\theta}\right)^{\tau}}\,dx = 3000\int_0^\infty e^{-y^3}\,dy = 3000\int_0^\infty \frac{e^{-a}}{3a^{\frac{2}{3}}}\,da = 1000\Gamma\left(\frac{1}{3}\right) = 2678.939$$

(b) With the policy limit, the expected loss is

$$\begin{aligned}\int_0^{5000} e^{-\left(\frac{x}{\theta}\right)^{\tau}}\,dx = 3000\int_0^{\frac{5}{3}} e^{-y^3}\,dy = 3000\int_0^{\left(\frac{5}{3}\right)^3} \frac{e^{-a}}{3a^{\frac{2}{3}}}\,da \\ = 1000\Gamma\left(\frac{1}{3}\right)P\left(X < \left(\frac{5}{3}\right)^3\right) = 2675.811\end{aligned}$$

(c) If there is 20% inflation, the new distribution is a Weibull distribution with $\theta = 3.6$ and $\tau = 3$, so the expected claim with the policy limit is

$$\begin{aligned}\int_0^{5000} e^{-\left(\frac{x}{\theta}\right)^{\tau}}\,dx = 3600\int_0^{\frac{25}{18}} e^{-y^3}\,dy = 3600\int_0^{\left(\frac{25}{18}\right)^3} \frac{e^{-a}}{3a^{\frac{2}{3}}}\,da \\ = 1200\Gamma\left(\frac{1}{3}\right)P\left(X < \left(\frac{25}{18}\right)^3\right) = 3179.07\end{aligned}$$

This is an increase of $\frac{3179.07}{2675.811} - 1 = 18.81\%$

89

For loss of \$6,000, the insurance pays \$4,000, so the insured pays \$2,000. The insured has paid \$10,000 when the total loss is \$14,000. The expected claim is therefore

$$0.8\int_{1000}^{6000}\frac{1}{\left(1+\frac{x}{5000}\right)^2}dx+0.9\int_{14000}\infty\frac{1}{\left(1+\frac{x}{5000}\right)^2}dx$$

Substituting $y = 5000 + x$, we get the expected claim amount is

$$\begin{aligned}
&0.8\int_{6000}^{11000}5000^2y^{-2}dy+0.9\int_{19000}^{\infty}5000^2y^{-2}dy=5000^2(0.8[-y^{-1}]_{6000}^{11000}+0.9[-y^{-1}]_{14000}^{\infty})\\
&=5000^2\left(\frac{0.8}{6000}-\frac{0.8}{11000}+\frac{0.9}{14000}\right)=\frac{308000-168000+297000}{462}=\frac{437000}{462}\\
&=\$945.89
\end{aligned}$$

90

The probability that a loss results in a payment is e^{-3}.

By the memoryless property of the exponential distribution, Y^P is exponentially distributed with mean 10000. It therefore has mean 10000 and variance 10000^2, so the coefficient of variation is 1.

$$\mathbb{E}(Y^L) = 10000 \int_3^\infty e^{-x} dx = 10000e^{-3}$$

and

$$\mathrm{Var}(Y^L) = 10000^2 e^{-3}(1 - e^{-3}) + e^{-3}10000^2 = 10000^2 e^{-3}(2 - e^{-3})$$

The coefficient of variation of the per-loss random variable is therefore

$$\frac{\sqrt{e^{-3}(2 - e^{-3})}}{e^{-3}} = e^{\frac{3}{2}} \sqrt{2 - e^{-3}} = 6.25868$$

2 9 Aggregate Loss Models

9.2 Model Choices

91

9.3 The Compound Model for Aggregate Claims

92 Let the moments of the primary distribution be μ_1, μ_2, μ_3 (and similar notation for raw moments). Let the moments of the secondary distribution be ν_1, ν_2, ν_3 (and similar notation for raw moments).

Recall that $P(z) = M(logz)$, so $P'(z) = \frac{M'(log(z))}{z}$, $P''(z) = \frac{M''(log(z))-M'(log(z))}{z^2}$, and $P'''(z) = \frac{M'''(log(z))-3M''(log(z))+2M'(log(z))}{z^3}$ In particular, $P'(1) = \mu$, $P''(1) = \mu_2' - \mu$ and $P'''(1) = \mu_3' - 3\mu_2' + 2\mu$.

m.g.f. of compound model is $P(M(z))$ first 3 derivatives of this at 0 are:

$$M'(0)P'(M(0)) = M'(0)P'(1) = \mu\nu$$
$$M''(0)P'(1) + M'(0)^2P''(1) = \mu\nu_2' + (\mu_2' - \mu)\nu^2$$
$$M'''(0)P'(1) + 3M''(0)M'(0)P''(1) + M'(0)^3P'''(1) = \mu\nu_3' + 3(\mu_2' - \mu)\nu\nu_2' + (\mu_3' - 3\mu_2' + 2\mu)\nu^3$$

93

For a given claim, the amount reimbursed has mean $1000 + 0.8 \times 500 = 1400$, and variance $500^2 + 0.8^2 \times 300^2 + 2 \times 0.8 \times 100000 = 467,600$.

The mean of the aggregate claims is therefore: $4 \times 1400 = 5600$. The variance is given by the law of total variance

$$\begin{aligned}
\mathrm{Var}(A) &= \mathbb{E}(N \,\mathrm{Var}(X_i)) + \mathrm{Var}(N\mathbb{E}(X_i)) \\
&= \mathbb{E}(N)\,\mathrm{Var}(X_i) + \mathbb{E}(X_i)^2\,\mathrm{Var}(N) \\
&= 4 \times 467600 + 1400^2 \times 4 \\
&= 9710400
\end{aligned}$$

Alternatively, the raw second moment is

$$4 \times (467600 + 1400^2) + (20 - 4) \times 1400^2 = 1,870,400 + 7,840,000 + 31,360,000 = 41,070,400$$

The variance is this minus 5600^2, which is $9,710,400$.

The standard deviation is the square root of this or $3,116.15$.

94

mean=$4 \times 6 \times 16 = 384$. Variance=$\mu\nu_2 + \mu_2\nu^2 = 6 \times 16 \times \frac{8^2}{12} + 4^2 \times 16 \times 6 \times 7 = 512 + 512 \times 21$ The standard deviation is therefore $32\sqrt{11}$. 95th percentile is 1.645 standard deviations above the mean or $384 + 52.64\sqrt{11} = 558.59$.

95

Prob of stop-loss is $e^{-1.25}$. Expected stop-loss claim conditional on claim is θ, so expected stop-loss claim=$e^{-1.25}\theta$. Premium is $2e^{-1.25}\theta$.

in fact value 0.9θ was used instead of theta, so premium is $1.8e^{-1.25}\theta$, and stop loss is really set at $1.25 \times 0.9\theta = 1.125\theta$, so expected payment on stop-loss is $e^{-1.125}\theta$. Percentage loading is therefore $\frac{1.8e^{-1.25}\theta}{e^{-1.125}\theta} - 1 = 1.8e^{-0.125} - 1 = 1.588494 - 1 = 58.85\%$.

9.8 Individual Risk Model

96

We will use a normal approximation for the first three types of workers, then treat the senior managers separately. We have the following for the first 3 types of workers:

Type of Worker	$\mathbb{E}(N)$	$\mathrm{Var}(N)$	$\mathbb{E}(S)$ (millions)	$\mathrm{Var}(S)$ $\times 10^{10}$
Manual Labourer	46.22	45.7578	4.622	45.7578
Administrator	7.08	7.06584	0.6372	5.7233304
Manager	8.02	7.9398	1.604	31.7592
Total			6.8632	83.2403304

Thus the aggregate losses for the first three groups can be approximated by a normal distribution with mean \$6,863,200 and standard deviation $\sqrt{832403304000} = \$912,361.388924$. We find the probability that the aggregate losses exceed \$10,000,000 by conditioning on the number of senior managers who die.

We can consider the various cases in a table

Senior Managers	Probability	Z-statistic	Probability aggregate more than 10,000,000	P
0	4.832131×10^{-01}	3.4381113	0.0002928934	1.415300×10^{-04}
1	3.550137×10^{-01}	2.3420544	0.0095889599	3.404212×10^{-03}
2	1.267906×10^{-01}	1.2459975	0.1063826587	1.348832×10^{-02}
3	2.932572×10^{-02}	0.1499406	0.4404057460	1.291522×10^{-02}
4	4.937494×10^{-03}	-0.9461163	0.8279553694	4.088025×10^{-03}
5	6.448972×10^{-04}	-2.0421732	0.9794328242	6.316334×10^{-04}
6	6.799936×10^{-05}	-3.1382301	0.9991501431	6.794157×10^{-05}
7	5.947466×10^{-06}	-4.2342870	0.9999885361	5.947398×10^{-06}
8	4.399911×10^{-07}	-5.3303439	0.9999999510	4.399911×10^{-07}
9	2.793594×10^{-08}	-6.4264008	0.9999999999	2.793594×10^{-08}
10	1.539327×10^{-09}	-7.5224577	1.0000000000	1.539327×10^{-09}
11	7.425327×10^{-11}	-8.6185146	1.0000000000	7.425327×10^{-11}
12	3.157027×10^{-12}	-9.7145715	1.0000000000	3.157027×10^{-12}
13	1.189461×10^{-13}	10.8106285	1.0000000000	1.189461×10^{-13}
14	3.987988×10^{-15}	11.9066854	1.0000000000	3.987988×10^{-15}
15	1.193683×10^{-16}	13.0027423	1.0000000000	1.193683×10^{-16}
16	3.197366×10^{-18}	14.0987992	1.0000000000	3.197366×10^{-18}
17	7.676750×10^{-20}	15.1948561	1.0000000000	7.676750×10^{-20}
18	1.653722×10^{-21}	16.2909130	1.0000000000	1.653722×10^{-21}
19	3.197313×10^{-23}	17.3869699	1.0000000000	3.197313×10^{-23}
20	5.546360×10^{-25}	18.4830268	1.0000000000	5.546360×10^{-25}
21	8.624078×10^{-27}	19.5790837	1.0000000000	8.624078×10^{-27}
22	1.200011×10^{-28}	20.6751406	1.0000000000	1.200011×10^{-28}
23	1.490697×10^{-30}	21.7711975	1.0000000000	1.490697×10^{-30}
24	1.647879×10^{-32}	22.8672544	1.0000000000	1.647879×10^{-32}
25	1.614249×10^{-34}	23.9633113	1.0000000000	1.614249×10^{-34}
26	1.393778×10^{-36}	25.0593682	1.0000000000	1.393778×10^{-36}
27	1.053498×10^{-38}	26.1554251	1.0000000000	1.053498×10^{-38}
28	6.910704×10^{-41}	27.2514820	1.0000000000	6.910704×10^{-41}
29	3.890614×10^{-43}	28.3475389	1.0000000000	3.890614×10^{-43}
30	1.852674×10^{-45}	29.4435958	1.0000000000	1.852674×10^{-45}
31	7.318000×10^{-48}	30.5396527	1.0000000000	7.318000×10^{-48}
32	2.333546×10^{-50}	31.6357096	1.0000000000	2.333546×10^{-50}
33	5.772531×10^{-53}	32.7317665	1.0000000000	5.772531×10^{-53}
34	1.039471×10^{-55}	33.8278234	1.0000000000	1.039471×10^{-55}
35	1.212212×10^{-58}	34.9238803	1.0000000000	1.212212×10^{-58}
36	6.871948×10^{-62}	36.0199372	1.0000000000	6.871948×10^{-62}

Total probability 0.0347433.

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The mean aggregate loss is $6863200 + 720000 = \$7,583,200$, and the variance of the aggregate loss is $832403304000 + 705600000000 = 1538003304000$, (so the standard deviation is 1240162.61192)

(a) Using a normal distribution, the probability that the aggregate loss exceeds 10,000,000 is $1-\Phi\left(\frac{10000000-7583200}{1240162.61192}\right) = 1-\Phi(1.94877669813) = 0.02566105$.

(b) Using a gamma distribution, we have $\theta = \frac{1538003304000}{7583200} = 202817.188522$ and $\alpha = \frac{7583200}{202817.188522} = 37.3893359594$ We are trying to calculate the probability that the distribution is more than 49.30548575573θ, which is given by $\frac{\int_{49.3054857573}^{\infty} x^{\alpha-1}e^{-x}\,dx}{\Gamma(\alpha)} = 0.03494672$

(c) Using a log-normal distribution, the mean of a log-normal distribution is $e^{\mu+\frac{\sigma^2}{2}}$, while the variance is $e^{2\mu+\sigma^2}(e^{\sigma^2}-1)$. We therefore have $e^{\sigma^2}-1 = \frac{1538003304000}{7583200^2} = 0.0267455940133$, so $\sigma^2 = \log(1.02674559401) = 0.0263941826449$. This gives $e^{\mu} = \frac{7583200}{\sqrt{1.02674559401}} = 7483781.279$ so $\mu = 15.8282487408$

The probability that this is greater than 10,000,000 is therefore $1-\Phi\left(\frac{\log(10000000)-15.8282487408}{\sqrt{0.0263941826449}}\right) = 1-\Phi(1.78408099245) = 0.03720525$

Type of Driver	$\mathbb{E}(N)$	$\mathrm{Var}(N)$	$\mathbb{E}(S)$ (thousands)	$\mathrm{Var}(S)$ (millions)
Safe	16	15.68	48	177.12
Average	105	99.75	420	1864.8
Unsafe	60	52.8	300	1455
			768	3496.92

(a) The gamma approximation therefore has $\theta = \frac{3496920000}{768000} = \frac{1165640}{256} = 4553.28125$ and $\alpha = \frac{768000}{4553.28125} = 168.669572080573$.

We get $\frac{800000}{\theta} = 175.697470917264$

The expected payment on the stop-loss insurance is therefore

$$\frac{\theta}{\Gamma(\alpha)}\int_{175.697470917264}^{\infty}(x^{\alpha} - 175.697470917264x^{\alpha-1})e^{-x}dx = \$11,234.2$$

The expected square of the payment on the stop-loss insurance is therefore

$$\frac{\theta^2}{\Gamma(\alpha)}\int_{175.697470917264}^{\infty}(x^{\alpha+1} - 2\times 175.697470917264x^{\alpha} + 175.697470917264^2x^{\alpha-1})e^{-x}dx = 740555835$$

so the variance of the stop-loss payment is 614348585, and the standard deviation is $24,786.06

The reinsurance premium is therefore $36,020.26.

(b) The normal approximation has $\mu = 768000$ and $\sigma^2 = 3496920000$, so the standard deviation is 59134.761350664128 and the cut-off for the stop-loss is 0.541136875656 standard deviations above the mean. The expected payment of the stop-loss is therefore

$$59134.761350664128\frac{\int_{0.541136875656}^{\infty}(x - 0.541136875656)e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$

$$= 59134.761350664128\left(\frac{[e^{-\frac{x^2}{2}}]_{0.541136875656}^{\infty}}{\sqrt{2\pi}} - 0.541136875656(1 - \Phi(0.541136875656))\right)$$

$$= 59134.761350664128\left(\frac{e^{-\frac{0.541136875656^2}{2}}}{\sqrt{2\pi}} - 0.541136875656(1 - \Phi(0.541136875656))\right) = 10963.59$$

The expected square of the payment is

$$59134.761350664128^2\frac{\int_{0.541136875656}^{\infty}(x-0.541136875656)^2e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$

$$=59134.761350664128^2\frac{\int_{0.541136875656}^{\infty}(x^2-1.082273751312x+0.292829118194)e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$

$$=\frac{59134.761350664128^2}{\sqrt{2\pi}}\left(\int_{0.541136875656}^{\infty}x\left(xe^{-\frac{x^2}{2}}\right)\,dx-1.082273751312\int_{0.541136875656}^{\infty}xe^{-\frac{x^2}{2}}\,dx\right.$$
$$\left.+0.292829118194\int_{0.541136875656}^{\infty}e^{-\frac{x^2}{2}}\,dx\right)$$

$$=\frac{59134.761350664128^2}{\sqrt{2\pi}}\left(\left[-xe^{-\frac{x^2}{2}}\right]_{0.541136875656}^{\infty}+\int_{0.541136875656}^{\infty}e^{-\frac{x^2}{2}}\,dx-1.082273751312\left[-e^{-\frac{x^2}{2}}\right]_{0.541136875656}^{\infty}\right.$$
$$\left.+\,0.292829118194\,(1-\Phi(0.541136875656))\right)$$

$$=59134.761350664128^2\left(1.292829118194\,(1-\Phi(0.541136875656))-\frac{0.541136875656}{\sqrt{2\pi}}e^{-\frac{0.541136875656^2}{2}}\right)$$

$$=677982110.383$$

So the variance is $677982110.383-10963.59^2=557781804.695$

The standard deviation is 23617.4046986 26038.088226267980, so the premium is $10963.59+23617.4046986=\$34,580.99$.

99

if 20% are smokers, the expected number of claims per policy is $0.2 \times 0.02 + 0.8 \times 0.01 = 0.012$, so the premium is set to $1.1 \times 12 = 1.32$. If 30% are smokers, the expected number of claims is per policy is 0.013. The variance of the number of claims is $0.3 \times 0.02 \times 0.98 + 0.7 \times 0.01 \times 0.99 = 0.01281$. The mean aggregate claim is therefore $13n$ and the variance of the aggregate claims is $12810n$. The total premium is $13.2n$. The probability that the total claims exceed total premiums is therefore $1 - \Phi\left(\frac{13.2n-13n}{\sqrt{12810n}}\right) < 0.2$. This means that $\frac{13.2n-13n}{\sqrt{12810n}} = 0.00176707682335\sqrt{n} > 0.8416212$ This means $n > \left(\frac{0.8416212}{0.00176707682335}\right)^2 = 226841.479733$.

So at least 226841 lives.

17 Introduction and Limited Fluctuation Credibility

17.2 Limited Fluctuation Credibility Theory

17.3 Full Credibility

100

(a) The number of claims made is a binomial distribution with $n = 372 \times 7 = 2604$ and some unknown p. The expected number of claims is np and the variance is $np(1-p)$, so the relative error $\frac{\overline{X}-\xi}{\xi}$ is approximately normally distributed with mean zero and variance $\frac{1-p}{np}$. We therefore want to check whether $\Phi\left(\frac{0.05}{\sqrt{\frac{1-p}{np}}}\right) \geqslant 0.975$ (two-sided confidence interval).

In this example, the total number of claims in seven years of experience is 9. This sets $p = \frac{9}{2604}$, and

$$\Phi\left(\frac{0.05}{\sqrt{\frac{1-p}{np}}}\right) = \Phi\left(\frac{0.15}{\sqrt{1-\frac{9}{2604}}}\right) = 0.5597202 < 0.975$$

So the company should not assign full credibility.

(b) Suppose we continue with the assumption that $p = \frac{9}{2604}$. Then we want to find the n such that

$$\Phi\left(\frac{0.05}{\sqrt{\frac{1-p}{np}}}\right) = \Phi\left(\frac{0.15\sqrt{n}}{\sqrt{2595}}\right) = 0.975$$

$$\frac{0.15\sqrt{n}}{\sqrt{2595}} = 1.96$$

$$n = \frac{1.96^2 \times 2595}{0.15^2} = 443064.5$$

If the company continues to employ 372 employees, then this equates to 1191.034 years.

101
Recall that we had

$$\begin{aligned}
\Phi\left(\frac{0.05}{\sqrt{\frac{1-p}{np}}}\right) &= 0.975 \\
0.05\sqrt{\frac{np}{1-p}} &= 1.96 \\
\frac{np}{1-p} &= 39.2^2 \\
np &= 1536.64(1-p) \\
(n+1536.64)p &= 1536.64 \\
p &= \frac{1536.64}{n+1536.64} \\
np &= \frac{1536.64n}{n+1536.64}
\end{aligned}$$

If p is small (and n is large), we can approximate $1-p=0$, so the standard for full credibility is 1568.64 claims. If n is smaller, then the standard for full credibility also gets smaller. For example, if $n = 1536.64$, then the standard for full credibility is only half as much.

102

(a)

Based on the data, the coefficient of variation is $\frac{3605.52}{962.14} = 3.747396$. Assuming the number of claims is large enough to use a normal approximation, we have that the critical value is 1.96 at the 95% confidence level. This means that the coefficient of variation for the average $\overline{X}$ is $\frac{3.747396}{\sqrt{41876}} = 0.01831247$. Multiplying by 1.96 gives us the relative 95% confidence interval as 0.03589244. Since this is less than 0.05, the company should assign full credibility to this data.

(b) The insurance company will assign full credibility if

$$\frac{3.747396}{\sqrt{n}} \times 1.96 \leqslant 0.05$$

$$n \geqslant \left(\frac{1.96 \times 3.747396}{0.05}\right)^2 = 21579$$

17.4 Partial Credibility

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The partial credibility assigned is $Z = \sqrt{\frac{7}{1191.034}} = 0.0766632$
The credibility premium is therefore

$$0.0766632 \times \frac{126000}{372} + 0.9233368 \times 1000 = \$949.303367742$$

Using 1568.64 claims as the standard for full credibility gives $Z = \sqrt{\frac{9}{1568.64}} = 0.075746$
The credibility premium is therefore

$$0.075746 \times 338.7097 + 0.924254 \times 1000 = \$949.91$$

104

(a) The credibility for claim frequency is $Z = \sqrt{\frac{19}{421}} = 0.2124397$, so the credibility estimate for claim frequency is $0.2124397 \times 1.9 + 0.7875603 \times 1.2 = 1.348708$.

The credibility for claim severity is $Z = \sqrt{\frac{19}{1240}} = 0.1237844$, so the credibility estimate for claim severity is $0.1237844 \times \frac{5822}{19} + 0.8762156 \times 230 = 239.4597$. The credibility estimate for aggregate claims is therefore $1.348708 \times 239.4597 = \$322.9613$.

(b) The credibility for claim frequency is $Z = \sqrt{\frac{19}{1146}} = 0.128761$, so the credibility estimate for claim frequency is $0.128761 \times 1.9 + 0.871239 \times 1.2 = 1.290133$.

The credibility for claim severity is $Z = \sqrt{\frac{19}{611}} = 0.1763422$, so the credibility estimate for claim severity is $0.1763422 \times \frac{5822}{19} + 0.8236578 \times 230 = 243.4763$. The credibility estimate for aggregate claims is therefore $1.290133 \times 243.4763 = \$314.1168$.

(c) The credibility for aggregate losses is $Z = \sqrt{\frac{10}{400}} = 0.1581139$. The credibility premium is therefore $0.1581139 \times 582.2 + 0.8418861 \times 276 = \$324.4145$.

(d) The credibility for aggregate losses is $Z = \sqrt{\frac{10}{1000}} = 0.1$. The credibility premium is therefore $0.1 \times 582.2 + 0.9 \times 276 = \$306.62$.

17.5 Problems with this Approach

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Using a normal approximation, the standard for full credibility is

$$\Phi\left(\frac{r\sqrt{n}}{\tau}\right) \geqslant 1-\frac{p}{2}$$

where τ is the coefficient of variation of X. For our data, we have

$$\tau = \frac{\sqrt{8240268} \times 3722}{3506608} = 3.046911$$

The standard for full credibility is therefore given by

$$\sqrt{n} = \frac{3.046911}{r}\left(\Phi^{-1}\left(1-\frac{p}{2}\right)\right)$$

The credibility is

$$Z = \sqrt{\frac{3722}{n}} = \frac{\sqrt{3722}}{3.046911\Phi^{-1}\left(1-\frac{p}{2}\right)}r = \frac{20.02297r}{\Phi^{-1}\left(1-\frac{p}{2}\right)}$$