ACSC/STAT 3703, Actuarial Models I

WINTER 2024
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Midterm Examination
Model Solutions

1. *A homeowner's house is valued at \$1,140,000 but is insured to a value of \$720,000. The insurer requires 80% coverage for full insurance. The home sustains \$9,000 of fire damage. The deductible is d, decreasing linearly to zero for losses above 2d. The insurer reimburses \$2,700. What is d?*

   The limit required for full insurance is $1140000 \times 0.8 = \$912,000$. The home therefore has $\frac{720000}{912000} = \frac{15}{19}$. This means that if the home were fully insured, the insurer would pay $\frac{19}{15} \times 2700 = 3420$. Thus, the deductible for a loss of \$9,000 is $9000 - 3420 = 5580$. This deductible is given by $d \times \frac{2d-9000}{d} = 2d - 9000$, so we have $2d - 9000 = 5580$, giving $d = \frac{9000+5580}{2} = 7290$.

   5 mins Cumulative Time: 5

2. *A home insurance company uses an expected loss ratio of 0.78. In accident year 2022, the earned premiums were \$34,400,000. For this accident year 2022, the insurance company made a total of \$9,140,000 in loss payments in 2022 and a total of \$11,630,000 in 2023. What loss reserves should the company hold for this accident year at the end of 2023, using the Expected Loss Ratio method?*

   The total loss payments made are \$20,770,000. The expected total payments are $34400000 \times 0.78 = 26832000$. Therefore the reserve should be $26832000 - 20770000 = \$6,062,000$.

   5 mins Cumulative Time: 10

3. *The following table shows the cumulative losses (in thousands) on claims from one line of business of an insurance company over the past 5 years.*

	Development year				
Accident year	*0*	*1*	*2*	*3*	*4*
2019	*1819*	*2659*	*4526*	*4452*	*4927*
2020	*1939*	*3011*	*4256*	*5180*	
2021	*2843*	*4388*	*7043*		
2022	*2376*	*3461*			
2023	*3071*				

The expected loss ratio is 0.78 and the earned premiums in each year are given in the following table:

Year	*Earned Premiums (000's)*
2019	*7319*
2020	*7972*
2021	*10528*
2022	*8989*
2023	*11748*

Using the mean for calculating loss development factors, esimate the total reserve needed for payments to be made in 2024 using the **chain-ladder method.** *[Assume that all claims from accident year 2019 have been finalised.]*

The mean loss development factors are

Development year	LDF
1/0	$\frac{13519}{8977} = 1.50595967472$
2/1	$\frac{15825}{10058} = 1.57337442832$
3/2	$\frac{9632}{8782} = 1.09678888636$
4/3	$\frac{4927}{4452} = 1.10669362084$

The estimated cumulative payments through 2024 are therefore:

Accident year	Expected cumulative payments
2020	$3071 \times 1.50595967472 = 4624.80216107$
2021	$3461 \times 1.57337442832 = 5445.44889642$
2022	$7043 \times 1.09678888636 = 7724.68412663$
2023	$5180 \times 1.10669362084 = 5732.67295595$
Total	23527.6081401

The total cumulative payments made through 2023 are $3071 + 3461 + 7043 + 5180 = 18755$. Thus the reserves for payments to be made in 2024 are $23527.6081401 - 18755 = 4772.6081401$.

15 mins Cumulative Time: 25

4. *In 2023, an auto insurer collected $14,800,000 in earned premiums, and paid $11,250,000 in payments. There was a rate change on 1st May 2022. Before the rate change, the premium was $920. After the rate change, the premium was $980. If inflation is 5%, what should the new premium be to achieve an expense ratio of 25% in policy year 2025? [Assume policies are sold at a uniform rate.]*

The old premium was sold for the first $\frac{4}{12}$ of 2022. Therefore, it applies to $\frac{1}{2} \times \left(\frac{4}{12}\right)^2 = \frac{1}{18}$ of the earned premiums in 2023. Therefore, if we adjust the earned premiums for 2023 to the new premium, we get $14800000 \times \frac{980}{\frac{1}{18}\times 920 + \frac{17}{18}\times 980} = 14850511.9454$. For these adjusted premiums, the loss ratio is $\frac{11250000}{14850511.9454} = 0.757549641478$. Therefore, the new premium without inflation is $\frac{0.757549641478}{0.75} \times 980 = 989.864864867$. Inflation in accident year 2023 is $\int_0^1 (1.05)^t\,dt = \frac{0.05}{\log(1.05)} = 1.02479671572$, and inflation in policy year 2025 is $\int_0^1 t(1.05)^t\,dt + \int_1^2 (2-t)(1.05)^t\,dt = \left(\frac{0.05}{\log(1.05)}\right)^2 = 1.05020830855$. Therefore, the new premium for policy year 2025 is $989.864864867 \times \frac{1.05020830855\times(1.05)^2}{1.02479671572} = \$1,118.39$.

15 mins Cumulative Time: 40

5. *The random variable T has moment generating function $M_T(t) = \frac{0.3}{(1-4t)(1-5t)} + \frac{0.7}{(1-6t)^3}$. Calculate the skewness of $(1.03)^T$.*

The raw moments of $(1.03)^T = e^{\log(1.03)T}$ are

$$\mathbb{E}((1.03)^T) = M_X(\log(1.03)) = \frac{0.3}{(1-4\log(1.03))(1-5\log(1.03))} + \frac{0.7}{(1-6\log(1.03))^3} = 1.65658359572$$
$$\mathbb{E}(((1.03)^T)^2) = M_X(2\log(1.03)) = \frac{0.3}{(1-8\log(1.03))(1-10\log(1.03))} + \frac{0.7}{(1-12\log(1.03))^3} = 3.16288738296$$
$$\mathbb{E}(((1.03)^T)^3) = M_X(3\log(1.03)) = \frac{0.3}{(1-12\log(1.03))(1-15\log(1.03))} + \frac{0.7}{(1-18\log(1.03))^3} = 7.66684202064$$

This means the centralised moments are

$$\mu_2 = \mu_2' - \mu_1^2 = 3.16288738296 - 1.65658359572^2 = 0.41861817335$$
$$\mu_3 = \mu_3' - 3\mu_1\mu_2' + 2\mu_1^3$$
$$= 7.66684202064 - 3 \times 3.16288738296 \times 1.65658359572 + 2 \times 1.65658359572^3 = 1.0403026692$$

Thus, the skewness is $\frac{1.0403026692}{0.41861817335^{1.5}} = 3.84089641924$.

10 mins Cumulative Time: 50

6. *Which distribution has a heavier tail: a distribution with hazard rate $\lambda(x) = 1 - e^{-x}$ or a distribution with survival function $S(x) = e^{-x^3}$? [Use any reasonable method for comparing tail-weight.]*

The easiest method is to look at the ratio of their hazard rate functions. For the distribution with $S(x) = e^{-x^3}$, we have $f(x) = 3x^2e^{-x^3}$, so $\lambda(x) = 3x^2$. Thus, the ratio of hazard rates is

$$\frac{3x^2}{1-e^{-x}}$$

This clearly tends to ∞ as $x \to \infty$, so the distribution with hazard rate $\lambda(x) = \frac{1-e^{-x}}{x}$ has a heavier tail.

Alternative solution:

For the distribution with hazard rate $\lambda(x) = 1 - e^{-x}$, the survival function is $S(x) = e^{-\int_0^\infty \lambda(x)\,dx} = e^{-x+1-e^{-x}}$, so the ratio of survival functions is $\frac{e^{-x+1-e^{-x}}}{e^{-x^3}} = e^{1-e^{-x}-x+x^3}$ which converges to ∞ as $x \to \infty$, so the distribution with hazard rate $\lambda(x) = 1 - e^{-x}$ has a heavier tail.

5 mins Cumulative Time: 55

7. *A measure of risk ρ satisfies all 4 coherence properties:*

- *Subadditivity*
- *Monotonicity*
- *Positive homogeneity*
- *Translation invariance*

If X follows an exponential distribution (gamma distribution with $\alpha = 1$) with mean $\theta = 4$ and Y follows a gamma distribution with $\alpha = 3$ and $\theta = 8$, then we have $\rho(X) = 4$ and $\rho(Y) = 23$. If Z follows a gamma distribution with $\alpha = 2$ and $\theta = 10$, which of the following is $\rho(Z)$?

(i) $\rho(Z) = 15.6$

(ii) $\rho(Z) = 17.9$

(iii) $\rho(Z) = 19.3$

(iv) $\rho(Z) = 22.1$

Justify your answer.

[Hint: If A follows a gamma distribution with parameters α and θ, then cA follows a gamma distribution with parameters α and $c\theta$. If A and B are independent and follow gamma distributions with parameters α_1 and θ and α_2 and θ respectively, then $A + B$ follows a gamma distribution with parameters $\alpha_1 + \alpha_2$ and θ.

Assume X and Z are independent, and let W be an exponential distribution with mean 4, independent of X.]

Let W have the same distribution as X, but be independent of X. Then $X + W$ follows a gamma distribution with $\alpha = 2$ and $\theta = 4$, so $2.5(X + W)$ has the same distribution as Z, so $\rho(Z) = \rho(2.5(X + W))$. By subadditivity and positive homogeneity, $\rho(Z) \leqslant 2.5\,(\rho(X) + \rho(W)) = 5\rho(X) = 20$.

Now suppose that Z is independent of X. Then $X + 0.4Z$ is a sum of independent gamma distributions with $\theta = 4$ and $\alpha = 1$ and $\alpha = 2$, so $X + 0.4Z$ follows a gamma distribution with $\alpha = 3$ and $\theta = 4$. Thus $2(X + 0.4Z)$ has the same distribution as Y. Thus $\rho(Y) \leqslant 2\rho(X) + 0.8\rho(Z)$, giving that $\rho(Z) \geqslant \frac{23-2\times 4}{0.8} = 18.75$.

Thus we have $18.75 \leqslant \rho(Z) \leqslant 20$, which is only satisfied by (iii) $\rho(Z) = 19.3$.

15 mins Cumulative Time: 70

8. *A distribution has support $\mathbb{R}^+$ and survival function given by $S(x) = \frac{250}{(x+5)^3} - \frac{8}{(x+2)^3}$. The VaR of this distribution at the 0.99 level is 23.79973756. Calculate the TVaR of this distribution at the* 0.99 *level.*

The TVaR is the conditional expectation given that the value is above the 99th percentile. Let $a = 23.79973756$ be the VaR at the 0.99 level. The TVaR is therefore given by

$$\begin{aligned}
\mathrm{TVaR}_{0.99}(X) &= a + \frac{1}{0.01}\int_a^\infty S(x)\,dx \\
&= a + 100\int_a^\infty \frac{250}{(x+5)^3} - \frac{8}{(x+2)^3}\,dx \\
&= a + \left[-\frac{12500}{(x+5)^2}\right]_a^\infty - \left[-\frac{400}{(x+2)^2}\right]_a^\infty \\
&= a + \frac{12500}{(a+5)^2} - \frac{400}{(a+2)^2} \\
&= 38.2694835219
\end{aligned}$$

10 mins Cumulative Time: 80