ACSC/STAT 3703, Actuarial Models I

WINTER 2023
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Practice Final Examination

Model Solutions

This Sample examination has more questions than the actual final, in order to cover a wider range of questions. Estimated times are provided after each question to help your preparation.

1. *A home insurance company uses an expected loss ratio of 0.77. In accident year 2020, the earned premiums were \$3,520,000. In 2020, the insurance company made a total of \$1,158,300 in loss payments for accident year 2020, a total of \$752,500 in 2021. At the end of 2022, the company sets the reserves for accident year 2020 to \$605,300. How much did the company pay in 2022 for losses in accident year 2020?*

   [5 mins]

   The expected total payments are $3520000 \times 0.77 = \$2,710,400$. In 2020 and 2021, the company paid a total of \$1,910,800. In order for the reserves to be \$605,300, the payments in 2022 must be $2710400 - 1910800 - 605300 = \$194,300$.

2. *The following table shows the cumulative losses (in thousands) on claims from one line of business of an insurance company over the past 4 years.*

	Development year			
Accident year	*0*	*1*	*2*	*3*
2019	*767*	*954*	*1388*	*1578*
2020	*1007*	*1388*	*1757*	
2021	*882*	*1082*		
2022	*1151*			

   Using the mean for calculating loss development factors, esimate the total reserve needed for payments to be made in 2024 using.

   (a) The loss development triangle method [15 mins]

   We calculate the following loss development factors:

Development year	Loss Development Factor
1/0	$\frac{954+1388+1082}{767+1007+882} = 1.28915662651$
2/1	$\frac{1388+1757}{954+1388} = 1.34286934244$
3/2	$\frac{1578}{1388} = 1.13688760807$

   Using the loss development triangle method, the cumulative payments up to 2023 are

$$1082 \times 1.34286934244 + 1151 \times 1.28915662651 = 2936.80390563$$

   The cumulative payments up to 2024 are

$$1082 \times 1.34286934244 \times 1.13688760807 + 1151 \times 1.28915662651 \times 1.34286934244 = 3644.45563583$$

The payments to be made in 2024 are therefore $3644.45563583 - 2936.80390563 = \$707.65$.

(b) The Bornhuetter-Fergusson method. The expected loss ratio is 0.82 and the earned premiums in each year are given in the following table:

Year	*Earned Premiums (000's)*
2019	*1940*
2020	*2594*
2021	*2104*
2022	*3194*

[15 mins]

Under the Bornhuetter-Fergusson method, the proportion of total payments in each year is given by:

Dev. Year	Proportion of total payments
0	$\frac{1}{1.13688760807\times1.34286934244\times1.28915662651} = 0.508092779138$
1	$\frac{1}{1.13688760807\times1.34286934244} - \frac{1}{1.13688760807\times1.34286934244\times1.28915662651} = 0.146918393967$
2	$\frac{1}{1.13688760807} - \frac{1}{1.13688760807\times1.34286934244} = 0.224583250215$
3	$1 - \frac{1}{1.13688760807} = 0.12040557668$

This gives us:

Year	Expected total losses	Expected losses in 2024
2021	1725.28	$1725.28 \times 0.12040557668 = 207.733333334$
2022	2619.08	$2619.08 \times 0.224583250215 = 588.201498973$

The total reserves needed for payments in 2024 are therefore $207.733333334 + 588.201498973 = \$795.93$.

3. *An insurance company collected a total of \$1,600,000 in premiums in 2022, and collects a total of \$2,200,000 in premiums in 2023. Assume the premium was constant throughout 2022 and 2023, and new policies were sold at constant rates in 2022 and in 2023 (but the rates in 2022 and 2023 are different). The estimated incurred losses for accident year 2023 are \$1,784,000. An actuary is using this data to estimate rates for premium year 2026. Claims are subject to 4% inflation per year. By what percentage should premiums increase from 2023 in order to achieve a loss ratio of 0.80? [15 mins]*

Half of the premiums collected in 2022 are earned in 2023, and half of the premiums collected in 2023 are earned that year, so the total earned premiums for 2022 are $\frac{1}{2}(1600000 + 2200000) = \$1,900,000$. The loss ratio is therefore $\frac{1784000}{1900000} = 0.938947368421$. Before inflation, the premium should be adjusted by a factor $\frac{0.938947368421}{0.8} = 1.17368421053$.

At the start of 2023, the number of premiums in force is proportional to 1600000, and at the end of 2023, it is propotional to 2200000, changing linearly throughout the year. Thus, the number of policies in force at time t during the year is proportional to $16+6t$. Thus the distribution of the time of a random loss has density $f(t) = \frac{16+6t}{19}$.

Thus, the expected inflation from the start of 2023 to the time of a random loss in 2023 is

$$\begin{aligned}\int_0^1 \frac{16+6t}{19}(1.04)^t\,dt &= \frac{16}{19}\frac{0.04}{\log(1.04)} + \frac{6}{19}\left(\left[t\frac{(1.04)^t}{\log(1.04)}\right]_0^1 - \int_0^1 \frac{(1.04)^t}{\log(1.04)}\,dt\right)\\ &= \frac{16}{19}\frac{0.04}{\log(1.04)} + \frac{6}{19}\left(\frac{1.04}{\log(1.04)} - \frac{0.04}{\log(1.04)^2}\right)\\ &= 1.02092187223\end{aligned}$$

The expected inflation from the start of 2026 to a random loss in policy year 2026 is $\left(\frac{0.04}{\log(1.04)}\right)^2 = 1.04013332308$. Thus, the premium should be adjusted by a factor of $\frac{1.17368421053\times(1.04)^3\times1.04013332308}{1.02092187223} = 1.34507896792$. This is an increase of 34.51%.

4. *In 2022, a home insurer collected $31,220,000 in earned premiums, and paid $24,030,000 in payments. There was a rate change on 1st September 2021. Before the rate change, the premium was $940. After the rate change, the premium was $980. Ignoring inflation, what should the new premium be to achieve an expense ratio of 20%? [15 mins]*

   The old premium was sold for the first $\frac{8}{12}$ of 2021. Therefore, it applies to $\frac{1}{2}\times\left(\frac{8}{12}\right)^2 = \frac{2}{9}$ of the earned premiums in 2022. Therefore, if we adjust the earned premiums for 2022 to the new premium, we get $31220000 \times \frac{980}{\frac{7}{9}\times980+\frac{2}{9}\times940} = \$31,505,766.5904$. For these adjusted premiums, the loss ratio is $\frac{24030000}{31505766.5904} = 0.762717514937$. Therefore, the new premium without inflation is $\frac{0.762717514937}{0.8}\times 980 = \$934.33$.

5. *An insurance company models claim sizes as following a mixture of two distributions. With probability 0.3, claims follow a Weibull distribution with $\tau = 2$ and $\theta = 100$. With probability 0.7, claims follow a Pareto distribution with $\alpha = 3$ and $\theta = 250$.*

   (a) Which of the following is the VaR of the distribution at the 90% level? [10 mins]

   (i) 122.02

   (ii) 146.35

   (iii) 197.14

   (iv) 230.60

   The survival function is

$$S(x) = 0.3e^{-\left(\frac{x}{100}\right)^2} + 0.7\frac{250^3}{(250+x)^3}$$

   .

   The VaR is the solution to

$$0.3e^{-\left(\frac{x}{100}\right)^2} + 0.7\frac{250^3}{(250+x)^3} = 0.1$$

   We try the values given:

x	$0.3e^{-\left(\frac{x}{100}\right)^2} + 0.7\frac{250^3}{(250+x)^3}$
(i) 122.02	0.2801187
(ii) 146.35	0.2108953
(iii) 197.14	0.128501
(iv) 230.60	0.1000009

so (iv) 230.60 is the VaR.

(b) Calculate the TVaR at the 90% level. [5 mins]

The TVaR is given as

$$230.60 + \frac{\int_{230.60}^{\infty} 0.3e^{-\left(\frac{x}{100}\right)^2} + 0.7\frac{250^3}{(250+x)^3}\,dx}{0.1}$$

We have

$$\int_{230.60}^{\infty} e^{-\frac{x^2}{10000}}\,dx = 100\sqrt{\pi}\left(1-\Phi\left(\frac{230.60\sqrt{2}}{100}\right)\right) = 100\sqrt{\pi}(1-\Phi(3.261176)) = 0.0983279$$

and

$$\int_{230.60}^{\infty} \frac{250^3}{(250+x)^3}\,dx = \left[-\frac{250^3}{2(250+x)^2}\right]_{230.60}^{\infty} = \frac{250^3}{2(250+230.60)^2} = 33.82381$$

so the TVaR is

$$230.60 + \frac{0.3\times 0.0983279 + 0.7\times 33.82381}{0.1} = 467.66$$

6. *An insurance company assigns a risk factor Θ to each individual. These Θ follow a gamma distribution with $\alpha = 2$ and $\theta = 400$. For an individual with risk factor $\Theta = \theta$, the size of a claim follows an inverse gamma distribution with $\alpha = 3$ and this value of θ. What is the probability that a random individual makes a claim in excess of \$3,000? [15 mins]*

   The probability that a claim from an individual with risk factor θ exceeds \$3,000 is

$$1 - e^{-\frac{\theta}{3000}}\left(1+\frac{\theta}{3000}+\frac{\theta^2}{2\times 3000^2}\right)$$

   Therefore the probability that a claim from a randomly chosen individual exceeds \$3,000 is the expected value of this over the distribution of Θ. That is,

$$\begin{aligned}
&1 - \int_0^{\infty} e^{-\frac{\theta}{3000}}\left(1+\frac{\theta}{3000}+\frac{\theta^2}{2\times 3000^2}\right)\frac{\theta}{400^2\Gamma(2)}e^{-\frac{\theta}{400}}\,d\theta\\
&= 1 - \int_0^{\infty} \frac{e^{-\frac{8.5\theta}{3000}}\left(\theta+\frac{\theta^2}{3000}+\frac{\theta^3}{2\times 3000^2}\right)}{400^2\Gamma(2)}\,d\theta\\
&= 1 - \frac{\left(\frac{3000}{8.5}\right)^2}{400^2} - \frac{2\left(\frac{3000}{8.5}\right)^3}{3000\times 400^2} - \frac{6\left(\frac{3000}{8.5}\right)^4}{2\times 3000^2\times 400^2}\\
&= 0.005938626
\end{aligned}$$

7. *The number of policies sold in a year follows a binomial distribution with $n = 100000$ and $p = 0.002$. The number of claims resulting from each policy sold follows a Poisson distribution with $\lambda = 0.02$. Calculate the variance of the total number of claims in a year. [10 mins]*

   We can use the law of total variance $\mathrm{Var}(S) = \mathbb{E}(\mathrm{Var}(S|X)) + \mathrm{Var}(\mathbb{E}(S|X))$ where X is the number of policies sold. We have $\mathrm{Var}(S|X) = 0.02X$, and $\mathbb{E}(S|X) = 0.02X$, so

$$\mathrm{Var}(S) = 0.02\mathbb{E}(X) + 0.0004\,\mathrm{Var}(X) = 0.02 \times 100000 \times 0.002 + 0.0004 \times 100000 \times 0.002 \times 0.998 = 4.0784$$

8. *A random variable X follows an extended truncated negative binomial distribution with $r = -0.4$ and $\beta = 0.2$. What is $P(X = 5)$? [5 mins]*

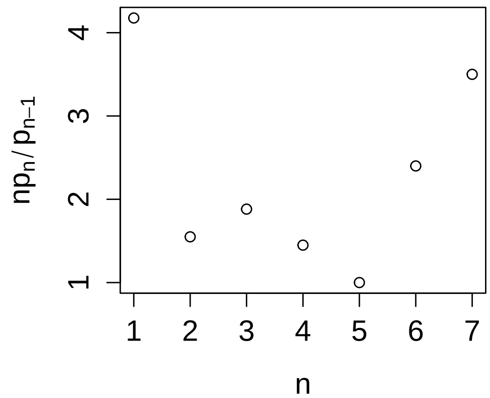
   For the truncated ETNB, we have $p_1 = \frac{r\beta}{(1+\beta)((1+\beta)^r-1)} = \frac{-0.08}{1.2(1.2^{-0.4}-1)} = 0.947874281065$. We also have $a = \frac{\beta}{(1+\beta)} = \frac{0.2}{1.2} = 0.166666666667$ and $b = \frac{(r-1)\beta}{1+\beta} = \frac{-1.4\times 0.2}{1.2} = -0.233333333333$. The recurrence then gives

$$\begin{aligned}
p_2 &= \frac{1}{6}\left(1 - \frac{1.4}{2}\right) \times 0.947874281065 = 0.0473937140533 \\
p_3 &= \frac{1}{6}\left(1 - \frac{1.4}{3}\right) \times 0.0473937140533 = 0.00421277458253 \\
p_4 &= \frac{1}{6}\left(1 - \frac{1.4}{4}\right) \times 0.00421277458253 = 0.000456383913108 \\
p_5 &= \frac{1}{6}\left(1 - \frac{1.4}{5}\right) \times 0.000456383913108 = 0.0000547660695731
\end{aligned}$$

9. *An insurer collects the following sample of claim frequencies.*

n	*Frequency*
0	*34*
1	*142*
2	*110*
3	*69*
4	*25*
5	*5*
6	*2*
7	*1*

   They make the following plot:



Which distribution(s) from the $(a, b, 0)$ or $(a, b, 1)$ classes might be suitable for modelling this data? Justify your answer. [5 mins]

We see that the points on the plot for $2 \leqslant n \leqslant 5$, the points follow a downward-sloping linear trend. The points for $n = 6$ and $n = 7$ are based on very small samples, so should not be given too much weight. This suggests distributions with $a < 0$, i.e. the binomial distribution would be appropriate. The point for $n = 5$ is also based on a relatively small sample, so it is conceivable that a distribution with $a = 0$, or a Poisson distribution might be suitable. In either case, the trend line through the points does not come close to passing through the point for $n = 1$, meaning that this should be modelled with a zero-modified distribution.

10. *The random variable X follows a distribution from the $(a, b, 1)$ class. We have $P(X = 3|X \geqslant 3) = \frac{2}{5}$, $P(X = 4|X \geqslant 4) = \frac{5}{12}$ and $P(X = 5|X \geqslant 5) = \frac{3}{7}$. What is $P(X = 6|X \geqslant 6)$? [10 mins]*

We have that $P(X = 3) = \frac{2}{5}P(X \geqslant 3)$, so $P(X \geqslant 4) = \frac{3}{5}P(X \geqslant 3)$, and therefore $P(X = 4) = \frac{5}{12} \times \frac{3}{5}P(X \geqslant 3) = \frac{5}{8}P(X = 3)$. This gives $a + \frac{b}{4} = \frac{5}{8}$.

Similarly, we have $P(X = 5) = \frac{\frac{3}{7}\times\frac{7}{12}}{\frac{5}{12}}P(X = 4)$, so $a + \frac{b}{5} = \frac{3}{5}$. This gives $a = 0.5$ and $b = 0.5$. Thus

$$P(X = 6) = \left(0.5 + \frac{0.5}{6}\right)P(X = 5) = \frac{7}{12} \times \frac{3}{7}P(X \geqslant 5)$$

and

$$P(X \geqslant 6) = P(X \geqslant 5) - P(X = 5) = \frac{4}{7}P(X \geqslant 5)$$

Thus

$$P(X = 6|X \geqslant 6) = \frac{1}{7}$$

[The information $P(X = 5|X \geqslant 5) = \frac{3}{7}$ can actually be calculated from the first two probabilities, but the algebra is more difficult and needs to be solved numerically.]

11. *The discrete random variable X satisfies the recurrence relation $P(X=n)=\left(a+\frac{b}{n+1}\right)P(X=n-1)$ for all $n\geqslant 1$. What is the probability generating function of X? [10 mins]*

We have $P(X+1=n)=\left(a+\frac{b}{n}\right)P(X+1=n-1)$ for all $n\geqslant 2$, so $X+1$ is a zero-truncated distribution from the $(a,b,1)$ class. Thus $X+1$ has p.g.f.

$$P(z)=\frac{(1-az)^{-\frac{a+b}{a}}-1}{(1-a)^{-\frac{a+b}{a}}-1}$$

. Thus X has p.g.f.

$$P(z)=\frac{(1-az)^{-\frac{a+b}{a}}-1}{z((1-a)^{-\frac{a+b}{a}}-1)}$$

12. *An insurance company models loss size as following a Pareto distribution with $\alpha=4$ and $\theta=6000$. The company introduces a deductible of \$1,000. Calculate the expected payment per claim after the deductible is introduced. [10 mins]*

The probability that a loss exceeds the deductible is $S(1000)=\left(\frac{6000}{6000+1000}\right)^4=0.5397751$. The survival function conditional on exceeding the deductible is

$$\frac{S(x)}{S(1000)}=\frac{\left(\frac{6000}{6000+x}\right)^4}{\left(\frac{6000}{7000}\right)^4}=\left(\frac{7000}{6000+x}\right)^4$$

so the loss per claim after the deductible is applied follows a Pareto distribution with $\alpha=4$ and $\theta=7000$. The expected payment per claim is therefore $\frac{7000}{4-1}=\$2,333.33$.

13. *An insurance company models loss size as following a Weibull distribution with $\tau=2$ and $\theta=2000$. The company wants to introduce a deductible so that the expected payment per loss is \$1400. What deductible should it introduce? [15 mins]*

With a deductible of d, the expected payment per loss is given by

$$\int_d^\infty S(x)\,dx=\int_d^\infty e^{-\left(\frac{x}{2000}\right)^2}\,dx=2000\sqrt{\pi}\int_d^\infty\frac{1}{\left(\sqrt{2\pi}\frac{2000}{\sqrt{2}}\right)}e^{-\frac{x^2}{2000^2}}\,dx$$
$$=2000\sqrt{\pi}\left(1-\Phi\left(\frac{d}{1000\sqrt{2}}\right)\right)$$

Setting this equal to \$1400 gives

$$
\begin{aligned}
2000\sqrt{\pi}\left(1-\Phi\left(\frac{d}{1000\sqrt{2}}\right)\right) &= 1400\\
\Phi\left(\frac{d}{1000\sqrt{2}}\right) &= 1-\frac{1400}{2000\sqrt{\pi}} = 0.6050673\\
\frac{d}{1000\sqrt{2}} &= 0.2664854\\
d &= 376.87
\end{aligned}
$$

14. *An insurance company models loss size as following a log-logistic distribution distribution with $\gamma = 2$ and $\theta = 2000$. The company wants to introduce a deductible with loss elimination ratio 30%.*

    (a) What deductible should it introduce? [15 mins]

    The mean of the log-logistic distribution is $2000\Gamma(1.5)\Gamma(0.5) = 1000\pi$, so to obtain a loss elimination ratio of 30%, the expected payment per loss after the deductible must be 700π.

    The expected payment per loss after a deductible of d is introduced is

$$
\begin{aligned}
\int_d^\infty S(x)\,dx = \int_d^\infty \frac{2000^2}{2000^2+x^2}\,dx &= 2000\int_{\frac{d}{2000}}^\infty \frac{1}{1+x^2}\,dx\\
&= 2000\left[\tan^{-1} x\right]_{\frac{d}{2000}}^\infty = 2000\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{d}{2000}\right)\right)
\end{aligned}
$$

    We therefore want to solve

$$
\begin{aligned}
2000\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{d}{2000}\right)\right) &= 700\pi\\
2000\tan^{-1}\left(\frac{d}{2000}\right) &= 1000\pi - 700\pi = 300\pi\\
\tan^{-1}\left(\frac{d}{2000}\right) &= \frac{300\pi}{2000} = 0.15\pi\\
\frac{d}{2000} &= \tan(0.15\pi)\\
d &= 2000\tan(0.15\pi) = 1019.05
\end{aligned}
$$

    (b) In the following years, there is uniform inflation of 4% every year. How many years does it take until the deductible calculated in (a) gives a loss elimination ratio of less than 25%? [15 mins]

If we discount for inflation, after n years, the deductible is equivalent to $1019.05(1.04)^{-n}$. We want to determine when this gives a loss elimination ration of 25%. As in part (a), a loss elimination ratio of 25% means the expected payment per loss is $\$750\pi$, so to get the deductible, we solve:

$$\begin{aligned}
2000\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{d}{2000}\right)\right) &= 750\pi \\
2000\tan^{-1}\left(\frac{d}{2000}\right) &= 1000\pi-750\pi=250\pi \\
\tan^{-1}\left(\frac{d}{2000}\right) &= \frac{250\pi}{2000}=0.125\pi \\
\frac{d}{2000} &= \tan(0.125\pi) \\
d &= 2000\tan(0.125\pi)=828.43
\end{aligned}$$

We therefore want to solve

$$\begin{aligned}
828.43 &= 1019.05(1.04)^{-n} \\
1.04^n &= \frac{1019.05}{828.43} \\
n &= \frac{\log\left(\frac{1019.05}{828.43}\right)}{\log(1.04)}=5.280325
\end{aligned}$$

So the loss elimination ratio is below 25% after 6 years.

15. *Losses follow a generalised Pareto distribution with $\alpha=2$, $\tau=3$, and $\theta=3000$. An insurance company introduces a deductible of \$600. Calculate the loss elimination ratio of this deductible after inflation of 12%. [15 mins]*

After inflation of 12%, losses follow a generalised Pareto distribution with $\alpha=2$, $\tau=3$ and $\theta=3360$. The expected loss is therefore $3360\frac{3}{2-1}=10080$. After the deductible, the expected payment per loss is

$$\begin{aligned}
\int_{600}^{\infty}(x-600)\frac{\Gamma(5)}{\Gamma(2)\Gamma(3)}\frac{\left(\frac{x}{3360}\right)^3}{x\left(1+\frac{x}{3360}\right)^5}\,dx &= 12\times3360^2\int_{600}^{\infty}\frac{x^3-600x^2}{(3360+x)^5}\,dx \\
&= 12\times3360^2\int_{3960}^{\infty}((u-3360)^3-600(u-3360)^2)u^{-5}\,du \\
&= 12\times3360^2\int_{3960}^{\infty}u^{-2}-10680u^{-3}+11280\times3360u^{-4}-3960\times3360^2u^{-5}\,du \\
&= 12\times3360^2\left[-u^{-1}+5340u^{-2}-3760\times3360u^{-3}+990\times3360^2u^{-4}\right]_{3390}^{\infty}\,du \\
&= 9482.087
\end{aligned}$$

The loss elimination ratio is therefore $1-\frac{9482.087}{10080}=\frac{597.91}{10080}=0.05931677$

16. *Losses follow an inverse Pareto distribution with $\tau = 4$ and $\theta = 6000$.*

(a) Calculate the expected payment per claim with a policy limit of $1,000,000. [15 mins]

The expected payment per claim is given by

$$
\begin{aligned}
&\int_0^{1000000} 1 - \left(\frac{x}{x+6000}\right)^4 dx \\
&= 1000000 - \int_0^{1000000} \left(\frac{x}{x+6000}\right)^4 dx \\
&= 1000000 - \int_{6000}^{1006000} \left(\frac{u-6000}{u}\right)^4 dx \\
&= 1000000 - \int_{6000}^{1006000} 1 - 24000u^{-1} + 6 \times 6000^2 u^{-2} - 4 \times 6000^3 u^{-3} + 6000^4 u^{-4}\, du \\
&= \left[24000\log(u) + 6 \times 6000^2 u^{-1} - 2 \times 6000^3 u^{-2} + \frac{6000^4}{3} u^{-3}\right]_{6000}^{1006000} \\
&= 24000\log\left(\frac{1006000}{6000}\right) + \frac{6 \times 6000^2}{1006000} - \frac{6 \times 6000^2}{6000} - \frac{2 \times 6000^3}{1006000^2} + \frac{2 \times 6000^3}{6000^2} + \frac{6000^4}{3 \times 1006000^3} - \frac{6000^4}{3 \times 6000^3} \\
&= \$97,141.75
\end{aligned}
$$

(b) Calculate the expected payment per claim if there is 15% inflation (the policy limit remains at $1,000,000.) [10 mins]

If there is 15% inflation, the loss distribution is an inverse Pareto distribution with $\tau = 4$ and $\theta = 6900$. The expected payment per claim is therefore

$$
\begin{aligned}
&\int_0^{1000000} 1 - \left(\frac{x}{x+6900}\right)^4 dx \\
&= 1000000 - \int_0^{1000000} \left(\frac{x}{x+6900}\right)^4 dx \\
&= 1000000 - \int_{6900}^{1006900} \left(\frac{u-6900}{u}\right)^4 dx \\
&= 1000000 - \int_{6900}^{1006900} 1 - 27600u^{-1} + 6 \times 6900^2 u^{-2} - 4 \times 6900^3 u^{-3} + 6900^4 u^{-4}\, du \\
&= \left[27600\log(u) + 6 \times 6900^2 u^{-1} - 2 \times 6900^3 u^{-2} + \frac{6900^4}{3} u^{-3}\right]_{6900}^{1006900} \\
&= 27600\log\left(\frac{1006900}{6900}\right) + \frac{6 \times 6900^2}{1006900} - \frac{6 \times 6900^2}{6900} - \frac{2 \times 6900^3}{1006900^2} + \frac{2 \times 6900^3}{6900^2} + \frac{6900^4}{3 \times 1006900^3} - \frac{6900^4}{3 \times 6900^3} \\
&= \$107,916.90
\end{aligned}
$$

17. *Losses follow an exponential distribution with $\theta = 7000$. There is a deductible of \$700, a policy limit of \$25,000 and coinsurance such that the insurance pays 80% of the claim after the policy limit and deductible have been applied. Calculate the expected payment per claim and the variance of the payment per claim. [15 mins]*

The expected payment per claim is

$$\begin{aligned}0.8\int_{700}^{25000}\frac{e^{-\frac{x}{7000}}}{e^{-\frac{700}{7000}}}\,dx &= 0.8\int_0^{24300} e^{-\frac{u}{7000}}\,du\\ &= 5600\left(1-e^{-\frac{24300}{7000}}\right) = \$5425.99\end{aligned}$$

The expected value of the square of the payment per claim is given by

$$\begin{aligned}0.64\int_0^{24300} 2xe^{-\frac{x}{7000}}\,dx &= 0.64\left([-14000xe^{-\frac{x}{7000}}]_0^{24300} + 14000\int_0^{24300} e^{-\frac{x}{7000}}\,dx\right)\\ &= 8960\left(7000(1-e^{-\frac{24300}{7000}}) - 24300e^{-\frac{24300}{7000}}\right) &= 54005749\end{aligned}$$

so the variance is $54005749 - 5425.99^2 = 24564349$

18. *Losses follow a Pareto distribution with $\alpha = 3$ and $\theta = 5000$. There is a deductible of \$1000. The insurance company wants to reduce the TVaR (per claim) for this policy at the 99.9% level to \$60,000. What policy limit should they set? [15 mins]*

The VaR at the 99.9% level is the solution to

$$\begin{aligned}\frac{\left(\frac{5000}{5000+(x+1000)}\right)^3}{\left(\frac{5000}{5000+1000}\right)^3} &= 0.001\\ \frac{5000}{5000+(x+1000)} &= 0.1\times\frac{5}{6}\\ \frac{x+1000}{5000}+1 &= 12\\ x &= 54000\end{aligned}$$

The TVaR with a policy limit of u is therefore

$$
\begin{aligned}
54000 &+ \frac{\int_{54000}^{u-1000}\left(\frac{6000}{5000+x+1000}\right)^3\,dx}{0.001}\\
&= 54000 + 1000\int_{60000}^{u+5000} 6000^3 s^{-3}\,ds\\
&= 54000 + 500[-6000^3 s^{-2}]_{60000}^{u+5000}\,ds\\
&= 54000 + \frac{500\times 6000^3}{60000^2} - \frac{500\times 6000^3}{(u+5000)^2}\\
&= 71361.11 - \frac{500\times 6000^3}{(u+5000)^2}
\end{aligned}
$$

The policy limit is therefore found by solving

$$
\begin{aligned}
71361.11 - \frac{500\times 6000^3}{(u+5000)^2} &= 60000\\
\frac{500\times 6000^3}{(u+5000)^2} &= 11361.11\\
(u+5000)^2 &= \frac{500\times 6000^3}{11361.11}\\
u &= \sqrt{\frac{500\times 6000^3}{11361.11}} - 5000 = \$92,499.30
\end{aligned}
$$

19. *Aggregate payments have a compund distribution. The frequency distribution is negative binomial with $r = 4$ and $\beta = 12$. The severity distribution is a Gamma distribution with $\alpha = 8$ and $\theta = 3000$. Use a normal approximation to aggregate payments to estimate the probability that aggregate payments are more than \$2,000,000. [15 mins]*

    The frequency distribution has mean 48 and variance 624. The severity distribution has mean 24000 and variance 72000000.

    The mean of aggregate payments is therefore, $48\times 24000 = 1152000$, and the variance is $624\times 24000^2 + 48\times 72000000 = 362880000000$, so the standard deviation is $\sqrt{362880000000} = 602395.2$. The probability of exceeding \$2,000,000 is therefore $1 - \Phi\left(\frac{2000000-1152000}{602395.2}\right) = 1 - \Phi(1.407714) = 1 - 0.9203921 = 0.0796$.

20. *Claim severity is modelled as following a Pareto distribution with $\alpha = 4.4$ and $\theta = 3,200$. A reinsurer offers stop-loss reinsurance with attachment point \$800,000 for a loading of 35%. Aggregate losses are modelled following a Pareto distribution with parameters fitted by matching moments. The resulting Pareto distribution has $\alpha = 4.2$. The reinsurance premium is \$18,000.*

    (a) What is the parameter θ for the Pareto distribution used to model aggregate losses? [15 mins]

(i) 483029

(ii) 614614

(iii) 703420

(iv) 783254

The expected payment on the stop-loss insurance is $\frac{18000}{1.35} = 13333.3333333$. For a Pareto distribution with parameters α and θ, the expected payment on a stop-loss insurance with attachment point a is

$$\begin{aligned}\int_a^\infty \left(\frac{\theta}{x+\theta}\right)^\alpha dx &= \int_{a+\theta}^\infty \theta^\alpha u^{-\alpha}\,du \\ &= \left[-\frac{\theta^\alpha u^{1-\alpha}}{(\alpha-1)}\right]_{a+\theta}^\infty \\ &= \frac{\theta^\alpha}{(\alpha-1)(a+\theta)^{\alpha-1}}\end{aligned}$$

Thus, we have that $\frac{\theta^\alpha}{(\alpha-1)(800000+\theta)^{\alpha-1}} = 13333.3333333$. Since $\alpha = 4.2$, we get $\frac{\theta^{4.2}}{3.2(800000+\theta)^{3.2}} = 13333.3333333$. We try the given values of θ:

θ	$\frac{\theta^{4.2}}{3.2(800000+\theta)^{3.2}} = 13333.3333333$
(i) 483029	6624.922
(ii) 614614	13333.337
(iii) 703420	19341.706
(iv) 783254	25744.184

We see that (ii) $\theta = 614614$ is the value of θ.

(b) What are the mean and variance of claim frequency? [15 mins]

The parameters of the Pareto distribution for modelling aggregate losses are $\alpha = 4.2$ and $\theta = 614614$. Since these were selected by matching moments, the mean of the aggregate loss distribution are $\frac{614614}{3.2} = 192066.875$ and $\frac{4.2\times 614614^2}{3.2^2\times 2.2} = 70425761265.1$.

Let μ and μ_2 be the mean and variance of the frequency distribution. The mean aggregate loss is $\frac{3200}{3.4}\mu = 941.176470588\mu = 192066.875$ which gives $\mu = 204.071054688$. matching the variance of aggregate loss gives

$$\begin{aligned}\left(\frac{3200}{3.4}\right)^2\mu_2 + \frac{4.4\times 3200}{3.4^2\times 2.4}\mu &= 70425761265.1 \\ \left(\frac{3200}{3.4}\right)^2\mu_2 &= 70425761265.1 - \frac{4.4\times 3200}{3.4^2\times 2.4}\times 204.071054688 = 70425657699.6 \\ \mu_2 &= 70425657699.6 \times \left(\frac{3.4}{3200}\right)^2 \\ &= 79503.9651374\end{aligned}$$

21. *An insurance company has the following portfolio of auto insurance policies:*

Type of driver	*Number*	*Probability claim*	*mean of claim*	*standard deviation*
Good driver	*60*	*0.02*	*\$2,500*	*\$14,000*
Average driver	*140*	*0.06*	*\$3,800*	*\$19,200*
Bad driver	*50*	*0.13*	*\$7,000*	*\$22,600*

Calculate the cost of reinsuring losses above \$500,000, if the loading on the reinsurance premium is one standard deviation above the expected claim payment on the reinsurance policy, using a Pareto approximation for the aggregate losses on this portfolio. [15 mins]

The expected aggregate loss is $60 \times 0.02 \times 2500 + 1400 \times 0.06 \times 380 + 50 \times 0.13 \times 7000 = 80420$, and the variance is $60 \times 0.02 \times 0.98 \times 2500^2 + 60 \times 0.02 \times 14000^2 + 140 \times 0.06 \times 0.94 \times 3800^2 + 140 \times 0.06 \times 19200^2 + 50 \times 0.13 \times 0.87 \times 7000^2 + 50 \times 0.13 \times 22600^2 = 7050179240$

We solve for the parameters of a Pareto distribution with these moments:

$$\begin{aligned}
\frac{\theta}{\alpha-1} &= 80420 \\
\frac{\theta^2\alpha}{(\alpha-1)^2(\alpha-2)} &= 7050179240 \\
\frac{\alpha-2}{\alpha} &= \frac{80420^2}{7050179240} = 0.91733503218 \\
\alpha &= \frac{2}{1-0.91733503218} = 24.1940455884 \\
\theta &= 23.1940455884 \times 80420 = 1865265.14622
\end{aligned}$$

The expected payment on the excess-of-loss reinsurance for losses above $a = 5000000$ is

$$\int_a^\infty \left(\frac{\theta}{\theta+x}\right)^\alpha dx = \int_{a+\theta}^\infty \theta^\alpha u^{-\alpha}\,du = \left[-\frac{\theta^\alpha u^{1-\alpha}}{\alpha-1}\right]_{a+\theta}^\infty = \frac{\theta^\alpha(a+\theta)^{1-\alpha}}{\alpha-1} = 325.947338998$$

where we have used the substitution $u = x + \theta$.

The expected square of the payment is

$$\begin{aligned}
\int_a^\infty 2(x-a)\left(\frac{\theta}{\theta+x}\right)^\alpha dx = \int_{a+\theta}^\infty 2(u-\theta-a)\theta^\alpha u^{-\alpha}\,du &= 2\left[-\frac{\theta^\alpha u^{2-\alpha}}{\alpha-2}\right]_{a+\theta}^\infty - 2\times 325.947338998(\theta+a) \\
&= 2\frac{\theta^\alpha(a+\theta)^{2-\alpha}}{\alpha-2} - 651.894677996(\theta+a) \\
&= 69473758.38
\end{aligned}$$

The variance is $69473758.38 - 325.947338998^2 = 69367516.7122$, so the mean plus one standard deviation is $325.947338998 + \sqrt{69367516.7122} = \$8,654.66$.

22. *An insurance company sells home insurance. It estimates that the standard deviation of the aggregate annual claim is \$5,326 and the mean is \$1,804.*

    (a) How many years history are needed for an individual or group to be assigned full credibility? (Use $r = 0.05$, $p = 0.95$.) [5 mins.]

    The variance of the mean of a sample of n observations from an individual is $\frac{5326^2}{n}$, so a 95% confidence interval for this individual is their mean plus or minus $1.96 \times \frac{5326}{\sqrt{n}}$. We want the relative error in this estimate to be at most 5%. That is we want

$$1.96 \times \frac{5326}{\sqrt{n}} = 0.05 \times 1804$$

$$n = \left(\frac{10438.96}{90.2}\right)^2 = 13393.73$$

    (b) What is the Credibility premium, using limited fluctuation credibility, for an individual who has claimed a total of \$42,381 in the past 19 years? [5 mins.]

    This individual's average annual aggregate claims are $\frac{42381}{19} = \$2230.58$. The credibility is $\sqrt{\frac{19}{13393.73}} = 0.03766396$, so the credibility premium is $0.03766396 \times 2230.58 + 0.96233604 \times 1804 = \$1820.07$.

23. *For a car insurance policy, the book premium for claim severity is \$2,300. An individual has made 7 claims in the past 12 years, with average claim severity \$1,074. Calculate the credibility estimate for claim severity for this individual using limited fluctuation credibility, if the standard for full credibility is:*

    (a) 157 claims. [5 mins.]

    If the standard for full credibility is 157 claims, then this individual's credibility is $\sqrt{\frac{7}{157}} = 0.2111539$, and the credibility estimate is $0.2111539 \times 1074 + 0.7888461 \times 2300 = 2041.13$.

    (b) 284 years. [5 mins.]

    If the standard for full credibility is 284 years, then this individual's credibility is $\sqrt{\frac{12}{284}} = 0.2055566$, and the credibility estimate is $0.2055566 \times 1074 + 0.7944434 \times 2300 = 2047.99$.