ACSC/STAT 3703, Actuarial Models I

WINTER 2024
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Homework Sheet 1

Model Solutions

Basic Questions

1. *A customer has utility function $u(x) = \log(x + 1000)$. The customer's current wealth is \$28,000. The customer's car has a value of \$14,700. The probability of the car being stolen is 0.016. How much would the customer be willing to pay for insurance against the car being stolen?*

   Without insurance, the customer has wealth \$28,000 if the car is not stolen (probability 0.984) and \$13,300 if the car is stolen (probability 0.016). The customer's expected utility without insurance is therefore $0.984 \log(29000) + 0.016 \log(14300) = 10.2637385284$. This is the same as the utility from wealth $e^{10.2637385284} - 1000 = 27673.7838159$. The customer would therefore be willing to pay up to $28000 - 27673.7838159 =$ \$326.22 for the insurance.

2. *Which of the following risks are insurable? For risks which are not insurable, explain why they are not insurable.*

   (i) The risk that a \$10 Christmas decoration will be broken.

   (ii) The risk that a borrower will need to pay interest on a debt.

   (iii) The risk that the interest rate on a debt will increase.

   (iv) The risk that an insurance company will have to pay too many losses.

   (v) The risk that an individual is late for an important meeting.

   (vi) The risk that a pregnancy will result in multiple births (twins, triplets, etc.) incurring unplanned expenses.

   (vii) The risk of an individual being killed by a malfunctioning self-flying aeroplane within the next 30 years.

   (viii) The risk that a dress will not be fashionable in two month's time.

   (i) This is not insurable as it is not economically feasible.

   (ii) This is not insurable as the loss is not random.

(iii) This is not insurable as losses from different policyholders are not independent. It might be insured via various financial derivatives. This is possible because, while losses are not independent, there is also a market for insuring against interest rates remaining too low, and these policies could be used as a hedge against the dependent risks.

(iv) This is an insurable risk — it is a reinsurance policy.

(v) This is not insurable as the risks are not homogeneous. It is also possible that the loss is not well-defined, leading to a severe risk of moral hazard.

(vi) This is an insurable risk, and indeed this coverage exists.

(vii) This is not insurable as there is not sufficient data to estimate the probability.

(viii) This is not insurable as the loss is not well-defined.

3. *A homeowner's house is insured at $950,000. The insurer requires 75% coverage for full insurance. The home sustains $25,300 damage from fire. The policy has a deductible of $10,000, which decreases linearly to zero when the total cost of the loss is $20,000. The house is valued at $1,360,000. How much does the insurer reimburse?*

   As the loss exceeds $20,000, there is no deductible. The home has $\frac{950000}{1360000\times 0.75} = 93.137254902\%$ coverage. This means that the insurer reimburses $25300 \times 0.93137254902 = \$23,563.73$.

4. *A liability insurance policy has a deductible of $100,000, a policy limit of $20,000,000 and co-insurance such that the policyholder pays 20% of the remaining claim. How much does the insurer pay if the loss is:*

   (i) $50,000

   (ii) $362,000

   (iii) $20,065,000

   (iv) $31,400,000

   (i) This is less than the deductible, so the insurer pays $0.

   (ii) $0.8(362000 - 100000) = \$209,600$.

   (iii) $0.8(20065000 - 100000) = \$15,972,000$.

   (iv) $0.8 \times 31400000 = 25120000$. As this exceeds the policy limit, the insurance pays the policy limit of $20,000,000.

Standard Questions

5. *An insurer charges a loading of 30% on its policies with limit \$500,000, and a loading of 33% on its policies with limit \$1,000,000. It purchases stop-loss reinsurance of \$500,000 over \$500,000. The cost of this reinsurance is 22% of total premiums. What is the reinsurer's loading on the reinsurance policy?*

   Let x be the expected losses limited to \$500,000 and let y be the expected losses limited to \$1,000,000. The insurer's premium for a policy with limit \$1,000,000 is $1.33y$, so the reinsurer's premium is $0.22 \times 1.33y = 0.2926y$. The insurer's premium for a policy with limit \$500,000 is $1.3x$, so the total premium for a policy with limit \$1,000,000 is $1.3x + 0.2926y = 1.33y$. This gives $1.3x = 1.0374y$, so $x = 0.798y$. The expected payment on the stop-loss policy is $y - x = 0.202y$, and the reinsurer's premium is $0.2926y$, so the reinsurer's loading is $\frac{0.2926}{0.202} - 1 = 44.85\%$.

6. *Policyholders are assumed to have a utility function $u(x) = -e^{-\frac{x}{\Theta}}$ where $\Theta > 0$ varies between policyholders following an exponential distribution with unknown mean. An insurance company sells an insurance policy which covers a risk which causes a loss of \$6,000 with probability 0.4. There are 3,000,000 potential customers for this policy. The insurer finds that when the premium for the policy is set to \$3000, they are able to sell 952,000 policies. How many policies would they sell if they increased the premium to \$4,000?*

   For a policyholder with wealth w and parameter Θ, the expected utility without insurance is

$$\mathbb{E}(u(x)) = -\left(0.6e^{-\frac{x}{\Theta}} + 0.4e^{-\frac{x-6000}{\Theta}}\right)$$
$$= -\left(0.6 + 0.4e^{\frac{6000}{\Theta}}\right)e^{-\frac{x}{\Theta}}$$

   If the policy has premium p, then a policyholder will buy it if

$$= -\left(0.6 + 0.4e^{\frac{6000}{\Theta}}\right)e^{-\frac{x}{\Theta}} < -e^{-\frac{x-p}{\Theta}}$$
$$= -e^{\frac{p}{\Theta}}e^{-\frac{x}{\Theta}}$$
$$0.6 + 0.4e^{\frac{6000}{\Theta}} > e^{\frac{p}{\Theta}}$$

   In particular for $p = 3000$, the policyholder will buy it if

$$\begin{aligned}
0.6+0.4e^{\frac{6000}{\Theta}} &> e^{\frac{3000}{\Theta}}\\
0.6+0.4\left(e^{\frac{3000}{\Theta}}\right)^2 &> e^{\frac{3000}{\Theta}}\\
2\left(e^{\frac{3000}{\Theta}}\right)^2-5e^{\frac{3000}{\Theta}}+3 &> 0\\
\left(2\left(e^{\frac{3000}{\Theta}}\right)-3\right)\left(e^{\frac{3000}{\Theta}}-1\right) &> 0\\
e^{\frac{3000}{\Theta}} &> 1.5\\
\Theta &< \frac{3000}{\log(1.5)} = 7398.91038713
\end{aligned}$$

[The second solution of the quadratic equation is satisfied by $\frac{3000}{\theta}=0$, which has no solutions.]

Similarly, for $p=4000$, the policyholder will buy it if

$$\begin{aligned}
0.6+0.4e^{\frac{6000}{\Theta}} &> e^{\frac{4000}{\Theta}}\\
0.6+0.4\left(e^{\frac{2000}{\Theta}}\right)^3 &> \left(e^{\frac{2000}{\Theta}}\right)^2\\
2\left(e^{\frac{2000}{\Theta}}\right)^3-5\left(e^{\frac{2000}{\Theta}}\right)^2+3 &> 0\\
\left(e^{\frac{2000}{\Theta}}-1\right)\left(2\left(e^{\frac{2000}{\Theta}}\right)^2-3\left(e^{\frac{2000}{\Theta}}\right)-3\right) &> 0\\
2\left(e^{\frac{2000}{\Theta}}\right)^2-3\left(e^{\frac{2000}{\Theta}}\right)-3 &> 0\\
\left(e^{\frac{2000}{\Theta}}\right) &> \frac{3+\sqrt{9+24}}{4} = 2.18614066164\\
\Theta &< \frac{2000}{\log(2.18614066164)} = 2557.094374
\end{aligned}$$

We also have that the probability of a policyholder buying the policy is $\frac{952000}{3000000}=0.317333333333$. Thus we have $P(\Theta > 7398.91038713) = 0.682666666667$, so the mean θ of the exponential distribution satisfies

$$\begin{aligned}
e^{-\frac{7398.91038713}{\theta}} &= 0.682666666667\\
\frac{7398.91038713}{\theta} &= -\log(0.682666666667) = 0.38174858149\\
\theta &= \frac{7398.91038713}{0.38174858149} = 19381.6316442
\end{aligned}$$

The probability that an individual buys the policy with premium \$4,000 is $P(\Theta < 2557.094374) = 1 - e^{-\frac{2557.094374}{19381.6316442}} = 0.12360108242$. Thus, they would sell $3000000 \times 0.12360108242 = 370803.24726$ policies.