ACSC/STAT 3703, Actuarial Models I

WINTER 2024
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Homework Sheet 2

Model Solutions

Basic Questions

1. *An insurer collects \$19,060,000 in earned premiums for accident year 2023. The total loss payments are \$15,329,000. Payments are subject to inflation of 4%, and policies are sold uniformly throughout the year. If the insurer's permissible loss ratio is 80%, by how much should the premium be changed for policy year 2026?*

The loss ratio in 2023 is $\frac{15329000}{19060000} = 0.804249737671$. Without inflation, the premium should be adjusted by a factor of $\frac{0.804249737671}{0.8} = 1.00531217209$. Inflation from the start of 2023 to a random claim in accident year 2023 is

$$\int_0^1 (1.04)^t\,dt = \left[\frac{(1.04)^t}{\log(1.04)}\right]_0^1 = \frac{0.04}{\log(1.04)} = 1.01986926764$$

Inflation from the start of 2026 to a random claim time for policy year 2026 is

$$\begin{aligned}
\int_0^1 t(1.04)^t\,dt + \int_1^2 (2-t)(1.04)^t\,dt &= \left(\frac{1.04}{\log(1.04)} - \frac{0.04}{\log(1.04)^2}\right) + 1.04\int_0^1 (1-t)(1.04)^t\,dt \\
&= \left(\frac{1.04}{\log(1.04)} - \frac{0.04}{\log(1.04)^2}\right) + 1.04\left(\int_0^1 1(1.04)^t\,dt - \int_0^1 t(1.04)^t\,dt\right) \\
&= 1.04\left(\frac{0.04}{\log(1.04}\right) - 0.04\left(\frac{1.04}{\log(1.04)} - \frac{0.04}{\log(1.04)^2}\right) \\
&= \frac{0.04^2}{\log(1.04)^2} \\
&= 1.04013332308
\end{aligned}$$

Therefore, the premium should be adjusted by a factor

$$\frac{1.00531217209 \times 1.04^3 \times 1.04013332308}{1.01986926764} = 1.15330842326$$

This is an increase of 15.33%.

2. *An insurer is reviewing claims for a certain line of insurance from Accident year 2023. The earned premiums in 2023 were \$7.7 million. The base premium in 2023 was \$1,120. However there was a rate change from the old premium of \$1,070 on 1st May 2023. The total losses in Accident Year 2023 were \$6.43 million. What should the new premium for Policy Year 2025 be if the permissible loss ratio is 0.75 and annual inflation is 7%?*

   [Assume policies are sold and losses occur uniformly through the year.]

   We first adjust the earned premiums to the current premium. The rate change happened 4 months into 2023, so the new premium applied to $\frac{1}{2}\times\left(\frac{8}{12}\right)^2=\frac{2}{9}$ of policy-years in accident year 2023. Therefore, the adjusted earned premiums are $7.7\times\frac{1120}{\frac{2}{9}\times 1120+\frac{7}{9}\times 1070}=7.97697841725$ million. The loss ratio is therefore $\frac{6.43}{7.97697841725}=0.806069624821$, so without inflation, the premiums should be adjusted by a factor $\frac{0.806069624821}{0.75}=1.07475949976$.

   Inflation from the start of 2023 to a random loss time in accident year 2023 is

$$\int_0^1 (1.07)^t\,dt=\left[\frac{1.07^t}{\log(1.07)}\right]_0^1=\frac{0.07}{\log(1.07)}=1.03460535466$$

   Inflation from the start of 2025 to a random loss time in Policy year 2025 is

$$\begin{aligned}\int_0^1 t(1.07)^t\,dt+\int_1^2(2-t)(1.07)^t\,dt &= \int_0^1 t(1.07)^t\,dt+1.07\int_0^1(1-t)(1.07)^t\,dt1.07\int_0^1(1.07)^t\,dt-0.07\int_0^1 t(1.07)^t\,dt\\ &=1.07\frac{0.07}{\log(1.07)}-0.07\left(\left[\frac{t1.07^t}{\log(1.07)}\right]_0^1-\int_0^1\frac{(1.07)^t}{\log(1.07)}\right)\\ &=\frac{0.07^2}{\log(1.07)^2}\\ &=1.07040823989\end{aligned}$$

   The premium for policy year 2025 is therefore

$$1120\times 1.07475949976\times 1.07^2\times\frac{1.07040823989}{1.03460535466}=\$1,425.84$$

3. *An insurance company has two lines of coverage in its auto insurance packages, with different expected loss ratios, and has the following data on recent claims:*

Policy Type	*Policy Year*	*Earned Premiums*	*Expected Loss Ratio*	*Losses paid to date*
	2021	*$16,000,000*	*0.78*	*$10,600,000*
Medical	*2022*	*$18,700,000*	*0.80*	*$6,300,000*
	2023	*$19,200,000*	*0.81*	*$3,900,000*
	2021	*$4,600,000*	*0.85*	*$3,500,000*
Property	*2022*	*$5,100,000*	*0.84*	*$3,400,000*
	2023	*$6,200,000*	*0.83*	*$2,800,000*

Calculate the loss reserves at the end of 2023.

We calculate the expected losses and the expected unpaid losses.

Policy Type	Policy Year	Expected total Losses	Losses paid to date	Reserves Needed
	2021	$12,480,000	$10,600,000	$1,880,000
Medical	2022	$14,960,000	$6,300,000	$8,660,000
	2023	$15,552,000	$3,900,000	$11,652,000
	2021	$3,910,000	$3,500,000	$410,000
Property	2022	$4,284,000	$3,400,000	$884,000
	2023	$5,146,000	$2,800,000	$2,346,000
Total				$25,832,000

So the total loss reserves needed at the end of 2023 are $25,832,000.

4. *The following table shows the cumulative paid losses (in thousands) on claims from one line of business of an insurance company over the past 5 years.*

Accident year	*Earned premiums*	*Development year 0*	*1*	*2*	*3*	*4*
2019	*37832*	*10873*	*16313*	*20489*	*23867*	*24452*
2020	*39619*	*8790*	*16279*	*23080*	*25669*	
2021	*44936*	*12498*	*20945*	*27821*		
2022	*47014*	*13098*	*21971*			
2023	*49669*	*11459*				

Assume that all payments on claims arising from accidents in 2019 have now been settled. Estimate the future payments arising each year from open claims arising from accidents in each calendar year using

(a) The loss development triangle method

First we compute the loss development factors:

0/1	$\frac{75508}{45259} = 1.66835325571$
1/2	$\frac{71390}{53537} = 1.33347031025$
2/3	$\frac{49536}{43569} = 1.13695517455$
3/4	$\frac{24452}{23867} = 1.02451083085$

Using these values to complete the table gives the following cumulative losses:

Accident year	Development year 0	1	2	3	4
LDF		1.66835325571	1.33347031025	1.13695517455	1.02451083085
2020				25669.00	26298.17
2021			27821.00	31631.23	32406.54
2022		21971.00	29297.68	33310.14	34126.60
2023	11459	19117.66	25492.83	28984.21	29694.63

The future payments are the differences between consecutive years:

Accident year	Development year 0	1	2	3	4
2019					629
2020				3810	775
2021			7327	4012	816
2022		7659	6375	3491	710

The total reserves needed are the sum of these, or 35606.

(b) The Bornhuetter-Ferguson method with expected loss ratio 0.81.

From the LDFs calculated in (a), we get the following proportions of losses paid.

Development Year	Cumulative proportion of losses paid	Proportion of losses paid
0	$\frac{1}{1.66835325571\times1.33347031025\times1.13695517455\times1.02451083085} = 0.385894634062$	0.385894634062
1	$\frac{1}{1.33347031025\times1.13695517455\times1.02451083085} = 0.643808569098$	0.257913935036
2	$\frac{1}{1.13695517455\times1.02451083085} = 0.858499612378$	0.21469104328
3	$\frac{1}{1.02451083085} = 0.976075576644$	0.117575964266
4	1	0.023924423356

This gives the following reserves:

Accident year	Earned premiums	Expected Total claims	Development year 0	1	2	3	4
2019	39619	32091.39					768
2020	44936	36398.16				4280	871
2021	47014	38081.34			8176	4477	911
2021	49669	40231.89		10376	8637	4730	963

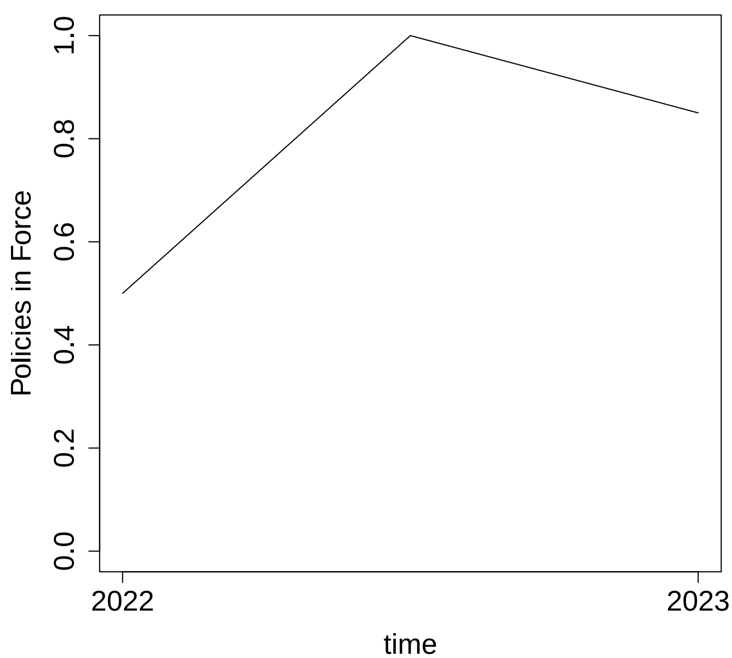
The total outstanding reserves are the sum of these payments, or 44189.

Standard Questions

5. *An insurance company starts a new line of insurance at the start of July 2021. It sells policies at a uniform rate throughout 2021, and for the first half of 2022. It then sells policies at a new rate that is 30% lower during the second half of 2022. It finds that by increasing its premium by 10%, it would have achieved the desired loss ratio for accident year 2022. The*

actuary estimates inflation is 5%. By how much should the premiums increase for policy year 2025, assuming policies are sold uniformly during 2024?

Let the rate of selling policies in the second half of 2021 and the first half of 2022 be r policies per year. The rate of selling policies in the second half of 2022 is $0.7r$, so the number of policies sold in 2022 is $\frac{1}{2}(r + 0.7r) = 0.85r$. The number of policies in force in the middle of 2022 is r, and the number of policies in force at the start of 2022 is $0.5r$. Between these times, the number of policies in force changes linearly.



Thus the average inflation in accident year 2022 is

$$\begin{aligned}
&\frac{1}{0.8375}\left(\int_0^{0.5}(0.5+t)(1.05)^t\,dt+\int_{0.5}^{1}(1.15-0.3t)(1.05)^t\,dt\right)\\
&\quad=\frac{80}{67}\left(0.5\left[\frac{1.05^t}{\log(1.05)}\right]_0^{0.5}+\left[t\frac{1.05^t}{\log(1.05)}\right]_0^{0.5}-\int_0^{0.5}\frac{1.05^t}{\log(1.05)}\,dt\right.\\
&\qquad\left.+1.15\left[\frac{1.05^t}{\log(1.05)}\right]_{0.5}^{1}-0.3\left[t\frac{1.05^t}{\log(1.05)}\right]_{0.5}^{1}+0.3\int_{0.5}^{1}\frac{1.05^t}{\log(1.05)}\,dt\right)\\
&\quad=\frac{80}{67}\left(\frac{0.5\left(1.05^{0.5}-1\right)}{\log(1.05)}+\frac{0.5(1.05)^{0.5}}{\log(1.05)}-\frac{1.05^{0.5}-1}{\log(1.05)^2}\right.\\
&\qquad\left.+\frac{1.15\left(1.05-1.05^{0.5}\right)}{\log(1.05)}-\frac{0.3\left(1.05-0.5(1.05^{0.5})\right)}{\log(1.05)}+\frac{0.3\left(1.05-1.05^{0.5}\right)}{\log(1.05)^2}\right)\\
&\quad=1.02652807692
\end{aligned}$$

The inflation from the start of 2025 to a random claim in policy year 2025 is $\frac{0.05^2}{\log(1.05)^2}=1.05020830854$.

Thus the premium needs to be increased by a factor $1.1\times1.05^2\times\frac{1.05020830854}{1.02652807692}=1.24072605009$ or 24.07%.

6. *An insurance company has the following cumulative aggregate loss development data:*

Accident year	*Development year* 0	*1*	*2*	*3*	*4*
2019	*2690*	*4084*	*8015*	*11636*	*15137*
2020	*3229*	*4359*	*8491*	*15104*	
2021	*3232*	*5514*	*8394*		
2022	*4026*	*6307*			
2023	*4095*				

From this table, it calculates the following mean loss development factors:

Development year	*LDF*
0/1	*1.537831*
1/2	*1.784051*
2/3	*1.620017*
3/4	*1.300877*

and the following cumulative reserves:

Accident year	*Development year* 0	*1*	*2*	*3*	*4*
2020					*19648.44*
2021				*13598.42*	*17689.87*
2022			*11252.010*	*18228.45*	*23712.96*
2023		*6297.418*	*11234.915*	*18200.75*	*23676.93*

It is discovered that one claim payment was recorded in the wrong year, and the cumulative losses for 2020 development year 1 should have been 5621. All other values remain the same.

(a) By how much do the necessary reserves at the end of 2023 decrease? [These are the total reserves for all expected payments after 2023 from all accident years.]

Changing the losses for 2020, development year 1 to 5621 changes the 0/1 LDF to $\frac{21527}{13177} = 1.63367989679$ and the 1/2 LDF to $\frac{24900}{15220} = 1.63600525624$. With these new LDFs, the total reserves needed for accident year 2023 are

$$4095 \times 1.63367989679 \times 1.63600525624 \times 1.620017 \times 1.300877 = 23065.4203035$$

the total reserves needed for accident year 2022 are

$$6307 \times 1.63600525624 \times 1.620017 \times 1.300877 = 21745.1963166$$

the total reserves needed for accident year 2021 are

$$8394 \times 1.620017 \times 1.300877 = 17689.8753241$$

and the total reserves needed for accident year 2020 are

$$15104 \times 1.300877 = 19648.446208$$

The total cumulative reserves plus payments already made for these years are therefore

$$23065.4203035 + 21745.1963166 + 17689.8753241 + 19648.446208 = 82148.9381522$$

The original cumulative reserves plus payments already made calculated for these years were

$$19648.44 + 17689.87 + 23712.96 + 23676.93 = 84728.2$$

Thus the reserves were decreased by 2579.3.

(b)

The earned premiums in each year are given in the following table:

Year	*Earned Premiums (000's)*
2018	*21,903*
2019	*22,743*
2020	*24,215*
2021	*24,886*
2022	*26,704*

Using the Bornhuetter-Fergusson method with expected loss ratio 0.81, the reserves for each year are:

Accident	*Expected*	*Development year*				
year	*Claims*	*0*	*1*	*2*	*3*	*4*
2020	*18421.83*					*4260.740*
2021	*19614.15*				*5770.553*	*4536.509*
2022	*20157.66*			*4203.601*	*5930.455*	*4662.216*
2023	*21630.24*		*2012.036*	*4510.687*	*6363.693*	*5002.806*

meaning that the total reserves are 47253.30. How much will the total reserves be changed if the cumulative losses for 2020, development year 1 are changed to 5621?

Replacing the LDFs for 0/1 and 1/2 by the corrected LDFs, 1.63367989679 and 1.63600525624, the proportion of total losses in each year is given by

Development Year	Cumulative proportion of losses paid	Proportion of losses paid
0	$\frac{1}{1.63367989679\times1.63600525624\times1.620017\times1.300877} = 0.177538494687$	0.177538494687
1	$\frac{1}{1.63600525624\times1.620017\times1.300877} = 0.290041069676$	0.112502574989
2	$\frac{1}{1.620017\times1.300877} = 0.474508714516$	0.18446764484
3	$\frac{1}{1.300877} = 0.768712184165$	0.294203469649
4	1	0.231287815835

Accident	Expected	Development year				
year	Claims	0	1	2	3	4
2020	18421.83		0.000	0.000	0.000	4260.745
2021	19614.15		0.000	0.000	5770.551	4536.514
2022	20157.66		0.000	3718.436	5930.454	4662.221
2023	21630.24		2433.458	3990.079	6363.692	5002.811

Thus, the total reserves needed are 46668.96. This is a reduction by 584.34.