ACSC/STAT 3703, Actuarial Models I

WINTER 2024
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Homework Sheet 3

Model Solutions

Basic Questions

1. *A distribution has hazard rate* $\lambda(x) = \frac{3}{2+x} + \frac{9}{4+x}$ *for* $x \geqslant 0$*. Calculate its survival function.*

   The survival function is

$$\begin{aligned} S(x) &= e^{-\int_0^x \lambda(s)\,ds} = e^{-\int_0^x \frac{3}{2+y}+\frac{9}{4+y}\,dy} \\ &= e^{-[3\log(2+y)+9\log(4+y)]_0^x} \\ &= e^{-(3\log(2+x)-3\log(2)+9\log(4+x)-9\log(4))} \\ &= \frac{2^{21}}{(2+x)^3(4+x)^9} \end{aligned}$$

2. *A continuous random variable has moment generating function given by* $M(t) = \frac{1}{(1-2t)^2(1-\theta t)^4}$ *for some parameter* $\theta > 0$*. What value of* θ *makes the coefficient of variation of the distribution equal to* $\frac{1}{4}$*?*

   We calculate

$$\begin{aligned} M'(t) &= \frac{4\theta}{(1-2t)^2(1-\theta t)^5} + \frac{4}{(1-2t)^3(1-\theta t)^4} \\ M''(t) &= \frac{20\theta^2}{(1-2t)^2(1-\theta t)^6} + \frac{32\theta}{(1-2t)^3(1-\theta t)^5} + \frac{24}{(1-2t)^4(1-\theta t)^4} \\ M'(0) &= 4\theta + 4 \\ M''(0) &= 20\theta^2 + 32\theta + 24 \end{aligned}$$

   Thus $\mathbb{E}(X) = 4\theta + 4$ and $\mathbb{E}(X^2) = 20\theta^2 + 32\theta + 24$. This gives $\mathrm{Var}(X) = 20\theta^2 + 32\theta + 24 - 16(\theta+1)^2 = 4\theta^2 + 8$. Thus, the coefficient of variation is $\frac{2\sqrt{\theta^2+2}}{4(\theta+1)}$, so we need to solve

$$\frac{2\sqrt{\theta^2+2}}{4(\theta+1)} = \frac{1}{4}$$
$$\frac{\theta^2+2}{4(\theta+1)^2} = \frac{1}{16}$$
$$4(\theta^2+2) = (\theta+1)^2$$
$$3\theta^2 - 2\theta + 7 = 0$$
$$\theta = \frac{2 \pm \sqrt{2^2 - 4\times 3\times 7}}{2\times 3}$$

So there is no real solution θ that has the required coefficient of variation.

3. *Calculate the mean excess loss function for a distribution with survival function given by $S(x) = 2e^{-\frac{x}{4}} - e^{-\frac{x}{2}}$ for $x \geqslant 0$.*

The mean excess loss function is given by

$$\begin{aligned}\mathbb{E}((X-d)_+) &= \int_d^\infty S(x)\,dx \\ &= \int_d^\infty 2e^{-\frac{x}{4}} - e^{-\frac{x}{2}}\,dx \\ &= \left[-8e^{-\frac{x}{4}} + 2e^{-\frac{x}{2}}\right]_d^\infty \\ &= 8e^{-\frac{d}{4}} - 2e^{-\frac{d}{2}}\end{aligned}$$

4. *Calculate the probability generating function of a discrete distribution with p.m.f. given by*

$$f(n) = \begin{cases} \frac{1+4e^{-4}+4e^{-8}}{9} & \text{if } n = 0 \\ \frac{4e^{-4}4^n+4e^{-8}8^n}{9(n!)} & \text{if } n > 0 \end{cases}$$

The probability generating function is given by

$$\begin{aligned}P(z) = \mathbb{E}(z^X) &= \sum_{n=0}^{\infty} f(x)z^x \\ &= \frac{1}{9}\left(1 + 4e^{-4}\sum_{n=0}^{\infty}\frac{4^n z^n}{n!} + 4e^{-8}\sum_{n=0}^{\infty}\frac{4^n z^n}{n!}\right) \\ &= \frac{1}{9}\left(1 + 4e^{-4}e^{4z} + 4e^{-8}e^{8z}\right) \\ &= \frac{1+4e^{4(z-1)}+4e^{8(z-1)}}{9}\end{aligned}$$

Standard Questions

5. *The total cost of handling a claim is $X+Y$ where X is a discrete non-negative random variable with probability generating function $P_X(z) = \left(\frac{1+2e^{-4(1-z)}}{3}\right)^3$ and Y is a continuous non-negative random variable. X and Y are independant. The moment generating function of the total cost is*

$$M_{X+Y}(t) = \frac{e^{2t}}{1-t}$$

. What is the moment generating function of Y?

Recall that

$$M_X(t) = \mathbb{E}\left(e^{tX}\right) = \mathbb{E}\left(\left(e^t\right)^X\right) = P_X\left(e^t\right) = \left(\frac{1+2e^{-4\left(1-e^t\right)}}{3}\right)^3$$

and that $M_{X+Y}(t) = M_X(t)M_Y(t)$ This gives,

$$M_Y(t) = \frac{M_{X+Y}(t)}{M_X(t)} = \frac{27e^{2t}}{(1-t)\left(1+2e^{-4(1-e^t)}\right)^3}$$

6. *An insurance company is trying to fit a log-logistic distribution to its claims data. The survival function for this distribution is given by*

$$S(x) = \frac{\theta^\gamma}{x^\gamma + \theta^\gamma}$$

The insurance company wants to select γ and θ so that the the 5th percentile and the 95th percentile match the observed values of 826 and 43,395 respectively. What values should they choose for γ and θ to achieve this?

We need to solve the equations

$$\frac{\theta^\gamma}{826^\gamma + \theta^\gamma} = 0.95$$
$$\frac{\theta^\gamma}{43395^\gamma + \theta^\gamma} = 0.05$$

$$\begin{aligned}
826^\gamma + \theta^\gamma &= \frac{20}{19}\theta^\gamma \\
826^\gamma &= \frac{\theta^\gamma}{19} \\
43395^\gamma + \theta^\gamma &= 20\theta^\gamma \\
43395^\gamma &= 19\theta^\gamma \\
\left(\frac{43395}{826}\right)^\gamma &= 361 \\
\gamma &= \frac{\log(361)}{\log\left(\frac{43395}{826}\right)} \\
&= 1.48652553914 \\
\theta &= 19^{\frac{1}{1.48652553914}} \times 826 \\
&= 5987.00843486
\end{aligned}$$