ACSC/STAT 3703, Actuarial Models I

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Homework Sheet 5

Model Solutions

Basic Questions

1. *A distribution of a random loss X in Canadian dollars has density function*

$$f_X(x) = \begin{cases} \frac{Ce^{-\frac{x^3}{5}}}{(x+1)^3} & \textit{if } x > 0 \\ \frac{4Ce^{\frac{x^3}{8}}}{(x-2)^2} & \textit{if} x \leqslant 0 \end{cases}$$

for some constant C. The loss is reinsured by a reinsurer in Europe, so needs to be converted to Euros using the conversion rate \$1 = €0.68304 *What is the density function for the loss in Euros?*

The density function is

$$f_{0.68304X}(x)\frac{1}{0.68304}f_X\left(\frac{x}{0.68304}\right) = \begin{cases} \frac{Ce^{-\frac{x^3}{1.59333984479}}}{0.68304(x+0.68304)^3} & \text{if } x > 0 \\ \frac{4Ce^{\frac{x^3}{2.54934375166}}}{0.68304(x-1.36608)^2} & \text{if} x \leqslant 0 \end{cases}$$

2. *Calculate the density function of X^5 when X follows a gamma distribution with $\alpha = 4$ and $\theta = 10$.*

The density function of X is $f_X(x) = \frac{x^3e^{-\frac{x}{\theta}}}{6\theta^4}$, so the distribution function of X^5 is

$$f_{X^5}(x) = 5x^{-\frac{4}{5}}f_X(x^{\frac{1}{5}}) = \frac{x^{-\frac{4}{5}}}{5}\frac{x^{\frac{3}{5}}e^{-\frac{x^{\frac{1}{5}}}{\theta}}}{6\theta^4} = \frac{x^{-0.2}e^{-\frac{x^{\frac{1}{5}}}{10}}}{300000}$$

3. *An individual's expected loss depends on the number of risk factors they have (with partial risk factors being possible). Each risk factor increases the expected loss by a factor 1.07. The number of risk factors follows a continuous distribution with moment generating function $M_T(t) = \frac{1225}{(5-t)^2(7-t)^2}$ What is the variance of an individual's expected loss?*

Let R be the number of risk factors. We have

$$\mathbb{E}\left(1.07^T\right) = \mathbb{E}\left(e^{\log(1.07)T}\right) = M_R(\log(1.07)) = \frac{1225}{(5-\log(1.07))^2\left(7-\log(1.07)\right)^2} = 1.04777965087$$

and

$$\mathbb{E}\left(\left(1.07^T\right)^2\right) = \mathbb{E}\left(e^{2\log(1.07)T}\right) = M_R(2\log(1.07)) = \frac{1225}{(5-2\log(1.07))^2\left(7-2\log(1.07)\right)^2} = 1.098464723$$

Therefore the variance of $(1.07)^R$ is

$$1.09846472389 - 1.04777965087^2 = 0.00062252711$$

4. *X is a mixture of 3 distributions:*

   - *With probability 0.2, X follows a Pareto distribution with $\alpha = 4.2$ and $\theta = 88$.*
   - *With probability 0.7, X follows a Weibull distribution with $\tau = 0.5$ and $\theta = 12$.*
   - *With probability 0.1, X follows a Gamma distribution with $\alpha = 2$ and $\theta = 5$.*

   The moments of these distributions are given in the following table:

	Distribution 1	*Distribution 2*	*Distribution 3*
Probability	*0.2*	*0.7*	*0.1*
μ	*27.5*	*24*	*10*
μ_2	*1443.75*	*2880*	*50*
μ_3	*344093.75*	*1022976*	*1250*
μ_4	*806866757.813*	*727584768*	*20000*
μ_2'	*2200*	*3456*	*75*
μ_3'	*484000*	*1244160*	*1500*
μ_4'	*851840000*	*836075520*	*75000*

   What is the skewness of X?

$$\begin{aligned}
\mathbb{E}(X) &= 0.2\times 27.5 + 0.7\times 24 + 0.1\times 10 = 23.3\\
\mathbb{E}(X^2) &= 0.2\times 2200 + 0.7\times 3456 + 0.1\times 75 = 2866.7\\
\mathbb{E}(X^3) &= 0.2\times 484000 + 0.7\times 1244160 + 0.1\times 1500 = 967862
\end{aligned}$$

This gives

$$\begin{aligned}
\mathbb{E}\left((X-\mathbb{E}(X))^2\right) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2\\
&= 2866.7 - 23.3^2 = 2323.81\\
\mathbb{E}\left((X-\mathbb{E}(X))^3\right) &= \mathbb{E}(X^3) - 3\mathbb{E}(X^2)\mathbb{E}(X) + 2\mathbb{E}(X)^3\\
&= 967862 - 3\times 2866.7\times 23.3 + 2\times 23.3^3 = 792778.344
\end{aligned}$$

so the skewness is $\frac{792778.344}{2323.81^{1.5}} = 7.07702702103$.

5. *For a particular claim, an insurance company has observed the following claim sizes:*

   1.9 2.6 2.9 3.3 4.7 6.9 11.8 20.2

   Using a kernel smoothing model with an exponential kernel (matching the mean of the kernel to the observed data), calculate the probability of a claim exceeding 25.

   For the kernel with mean x_i, the probability that the claim exceeds 25 is $e^{-\frac{25}{xi}}$.

x_i	$e^{-\frac{25}{x_i}}$
1.9	0.00000193018493808
2.6	0.0000666947302187
2.9	0.000180335842259
3.3	0.000512731841409
4.7	0.0048969195434
6.9	0.0266974184424
11.8	0.120194493654
20.2	0.290072681257
Total	0.442623205496

   Thus, the probability is $\frac{0.442623205496}{8} = 0.055327900687$.

Standard Questions

6. *An insurance company models the claims of an individual (in dollars) as following an inverse Pareto distribution with $\theta = 100$ and τ varying between individuals. For a random individual, τ is assumed to follow a gamma distribution with shape parameter $\alpha = 0.5$ and scale parameter θ.*

   From the insurer's data, 1.2% of claims exceed \$1,000. What percentage of claims exceed \$100,000?

   For the inverse Pareto distribution with $\theta = 100$ and τ, the probability that a claim is less than \$1,000 is $\left(\frac{100}{100+1000}\right)^{\tau}$. Therefore, the overall probability that a random claim exceeds \$1,000 is

$$\mathbb{E}\left(11^{-\tau}\right) = M_{\tau}(-\log(11))$$

   The moment generating function of the gamma distribution is $M(t) =$

$(1-\theta t)^{-\alpha}$ so

$$\begin{aligned}0.988 &= M_\tau(-\log(1.1)) = (1+\theta\log(1.1))^{-0.5}\\ \theta &= \frac{\frac{1}{0.988^2}-1}{\log(1.1)}\\ &= 0.256415602616\end{aligned}$$

For a fixed value of τ, the probability that a loss exceeds \$100,000 is $1-\left(\frac{100000}{100100}\right)^\tau$, so the probability that a random loss exceeds \$100,000 is

$$1-\mathbb{E}(1.001^{-\tau}) = 1-M_\tau(-\log(1.001)) = 1-(1+0.256415602616\log(1.001))^{-0.5} = 0.0128119113\%$$

7. *The time until failure of a product has hazard rate $\lambda(t) = \frac{a}{\sqrt{t}} + 0.1\sqrt{t}$ where a is the probability that the product is defective, and is modelled as following a distribution with moment generating function $M_A(t) = e^{e^{\frac{\sqrt{t}}{20}}-1}$. Given that a product has lasted for one year waranty, what is the probability that it will be last another year?*

   [Note: the given formula for the moment generating function is incorrect — giving complex values for negative numbers.]

   For a given value of a, the probability that the product will last for s years is

   $$e^{-\int_0^s \frac{a}{\sqrt{t}}+0.1\sqrt{t}\,dt} = e^{-\left[2a\sqrt{t}-\frac{t^{1.5}}{15}\right]_0^s} = e^{-2\sqrt{s}a-\frac{s^{1.5}}{15}}$$

   The probability that a random product will last for s years is therefore

   $$e^{-\frac{s^{1.5}}{15}}\mathbb{E}\left(e^{-2\sqrt{s}A}\right) = e^{-\frac{s^{1.5}}{15}}M_A(-2\sqrt{s})$$

   In particular, the probability of lasting for 1 year is $e^{-\frac{1}{15}}e^{e^{\frac{\sqrt{-2}}{20}}-1} = 0.9308440345-0.0658754221i$. The probability of lasting for 2 years is $e^{-\frac{2^{1.5}}{15}}e^{e^{\frac{\sqrt{-2\sqrt{2}}}{20}}-1} = 0.8117915119-0.1151861712i$. The probability that a product that has already lasted for 1 year lasts for another year is therefore $\frac{0.8117915119-0.1151861712i}{0.9308440345-0.0658754221i} = 0.8764702509-0.0617163792i$.