ACSC/STAT 3703, Actuarial Models I

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Homework Sheet 6

Model Solutions

Basic Questions

1. *Let X follow a negative binomial distribution with $r = 4.1$ and $\beta = 1.3$. What is the probability that $X = 4$?*

   The probability is

$$P(X=4) = \binom{3.1+4}{4}\frac{1}{2.3^{4.1}}\left(\frac{1.3}{2.3}\right)^4 = 0.12662192518$$

2. *The number of claims on each insurance policy over a given time period is observed as follows:*

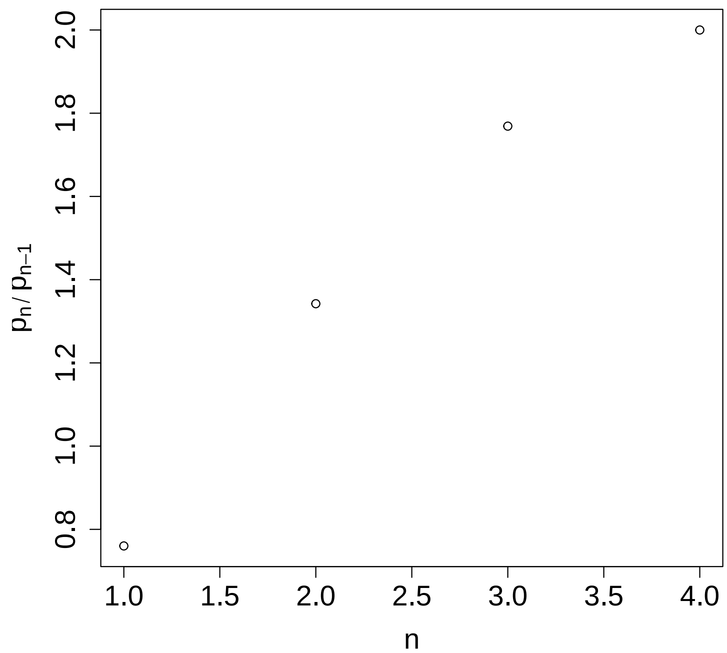
Number of claims	*Number of policies*
0	*492*
1	*374*
2	*251*
3	*148*
4	*74*
5 or more	*92*

   Which distribution(s) from the $(a, b, 0)$-class and $(a, b, 1)$-class appear most appropriate for modelling this data?

   We estimate

n	$\frac{p_n}{p_{n-1}}$
1	$\frac{374}{492} = 0.760162601626$
2	$\frac{251}{374} = 0.671122994652$
3	$\frac{148}{251} = 0.589641434263$
4	$\frac{74}{148} = 0.5$

   We can plot a graph of $n\frac{p_n}{p_{n-1}}$ against n. For a distribution from the $(a, b, 0)$ class, this should be linear with slope a and intercept b. For a distribution from the $(a, b, 1)$ class, all points for $n \neq 1$ should be linear.



On this graph, the slope is positive, so $a > 0$. The points appear to all be approximately linear, so there is not strong evidence for zero modification. This suggests a negative binomial distribution.

3. *X follows an extended modified negative binomial distribution with $r = -0.7$ and $\beta = 2.3$, and $p_0 = 0.7$. What is $P(X = 3)$?*

For the truncated ETNB with $r = -0.7$ and $\beta = 2.3$, we have $p_1 = \frac{r\beta}{(1+\beta)((1+\beta)^r-1)} = \frac{-0.7\times 2.3}{3.3(3.3^{-0.7}-1)} = 0.861293480447$ We also have $a = \frac{\beta}{1+\beta} = \frac{2.3}{3.3} = 0.69696969697$ and $b = (r-1)a = -1.7\times 0.69696969697 = -1.18484848485$.

$$p_2 = \left(0.69696969697 - \frac{1.18484848485}{2}\right) \times 0.861293480447 = 0.09004431841$$

$$p_3 = \left(0.69696969697 - \frac{1.18484848485}{3}\right) \times 0.09004431841 = 0.0271952032369$$

Now for the distribution with $p_0 = 0.7$, we have $P(X = 3) = 0.0271952032369 \times 0.3 = 0.00815856097107$.

4. *Let X follow a mixed zero-modified Poisson distribution with $\lambda = 3.7$, where p_0 follows a beta distribution with $\alpha = 3$ and $\beta = 8$. What is the*

probability that $X = 2$?

For a zero-truncated Poisson distribution, $p_2 = \frac{3.7^2e^{-3.7}}{2(1-e^{-3.7})} = 0.173522630026$. For a given value of p_0, this probability is therefore $0.173522630026(1-p_0)$. The marginal probability that $X = 2$ is therefore $\mathbb{E}\left(0.173522630026(1-P_0)\right)$. For the beta distribution with $\alpha = 3$ and $\beta = 8$, $\mathbb{E}(P_0) = \frac{3}{11}$, so $P(X = 2) = 0.173522630026\left(1 - \frac{3}{11}\right) = 0.126198276383$.

Standard Questions

5. *We can find the mean of a distribution from the $(a, b, 1)$-class as follows:*

$$\mathbb{E}(X) = \sum_{n=1}^{\infty} np_n = p_1 + \sum_{n=2}^{\infty} n\left(a + \frac{b}{n}\right)p_{n-1} = p_1 + \sum_{m=2}^{\infty}(a(m+1)+b)p_m = p_1 + a\mathbb{E}(X) + (a+b)(1-p_0)$$

Thus $\mathbb{E}(X) = \frac{p_1+(a+b)(1-p_0)}{1-a}$ Use similar methods to find the raw second moment, and hence determine the value of p_0 that maximises variance for a general distribution from the $(a, b, 1)$-class.

[Hint: for a zero truncated distribution from the $(a, b, 1)$-class, the probability of 1 is $p_1^T = \frac{a+b}{(1-a)^{-1-\frac{b}{a}}-1}$.]

We compute the raw second moment as follows:

$$\begin{aligned}
\mathbb{E}(X^2) &= \sum_{n=1}^{\infty} n^2 p_n \\
&= p_1 + \sum_{n=2}^{\infty} n^2\left(a + \frac{b}{n}\right)p_{n-1} \\
&= p_1 + \sum_{m=1}^{\infty}(a(m+1)^2 + b(m+1))p_m \\
&= p_1 + a\mathbb{E}(X^2) + (2a+b)\mathbb{E}(X) + (a+b)(1-p_0) \\
&= p_1 + a\mathbb{E}(X^2) + (2a+b)\frac{p_1+(a+b)(1-p_0)}{1-a} + (a+b)(1-p_0) \\
(1-a)\mathbb{E}(X^2) &= \frac{(1+a+b)p_1}{1-a} + \frac{(a+b)(1+a+b)(1-p_0)}{1-a} \\
\mathbb{E}(X^2) &= \frac{(1+a+b)p_1 + (a+b)(1+a+b)(1-p_0)}{(1-a)^2}
\end{aligned}$$

We then get

$$\begin{aligned}\mathrm{Var}(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\ &= \frac{(1+a+b)p_1 + (a+b)(1+a+b)(1-p_0)}{(1-a)^2} - \left(\frac{p_1 + (a+b)(1-p_0)}{1-a}\right)^2 \\ &= \frac{(1+a+b-p_1)p_1 + (a+b)(1-p_0) - 2(a+b)p_1(1-p_0)}{(1-a)^2}\end{aligned}$$

Recall that, if we let p_1^T be the probability of 1 for a zero-truncated random variable, then $p_1 = p_1^T(1-p_0)$, so

$$\mathrm{Var}(X) = \frac{(1+a+b-p_1^T(1-p_0))p_1^T(1-p_0) + (a+b)(1-p_0) - 2(a+b)p_1^T(1-p_0)^2}{(1-a)^2}$$

letting $q_0 = (1-p_0)$, we can rewrite this as

$$\mathrm{Var}(X) = \frac{((1+a+b)p_1^T + a + b)q_0 - ((p_1^T)^2 + 2(a+b)p_1^T)q_0^2}{(1-a)^2}$$

It is straighforward to see that this is maximised by

$$q_0 = \frac{(1+a+b)p_1^T + a + b}{2p_1^T(p_1^T + 2(a+b))}$$

Substituting $p_1^T = \frac{a+b}{(1-a)^{-1-\frac{b}{a}} - 1}$ gives

$$q_0 = \frac{\frac{(1+a+b)}{(1-a)^{-1-\frac{b}{a}}-1} + 1}{\frac{2(a+b)}{(1-a)^{-1-\frac{b}{a}}-1}\left(\frac{1}{(1-a)^{-1-\frac{b}{a}}-1} + 2\right)} = \frac{a+b+(1-a)^{-1-\frac{b}{a}}}{\left(2(a+b)\frac{2(1-a)^{-1-\frac{b}{a}}-1}{(1-a)^{-1-\frac{b}{a}}-1}\right)}$$

whenever this is less than 1, and when this is more than 1, variance is maximised by $p_0 = 0$.

6. *A random variable X is assumed to have distribution in the $(a, b, 1)$-class. An insurance company collects the following sample of truncated X values:*

Value	*Frequency*
$X = 5$	*1256*
$X = 6$	*875*
$X = 7$	*590*

Assuming that the probabilites are proportional to these values, If the company observed all values from the sample, how many values would they expect to observe with $X = 1$?

We have $a+\frac{b}{6} = \frac{875.0}{1256} = 0.696656050955$ and $a+\frac{b}{7} = \frac{590}{875} = 0.674285714286$. This gives

$$
\begin{aligned}
6a + b &= 4.17993630573 \\
7a + b &= 4.72 \\
a &= 0.54006369427 \\
b &= 0.93955414011
\end{aligned}
$$

This gives $n_1 = \frac{1256}{\left(0.54006369427+\frac{0.93955414011}{5}\right)\left(0.54006369427+\frac{0.93955414011}{4}\right)\left(0.54006369427+\frac{0.93955414011}{3}\right)\left(0.54006369427+\right.}$ 2583.86692639.