ACSC/STAT 3703, Actuarial Models I

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Homework Sheet 7

Model Solutions

Basic Questions

1. *An insurance company has an insurance policy where the loss amount follows a Gamma distribution with $\alpha = 3$ and $\theta = 400$. Calculate the expected payment per claim if the company introduces a deductible of d.*

For the Gamma distribution $f(x) = \frac{x^2 e^{-\frac{x}{400}}}{2\times 400^3}$. The expected payment per loss is

$$\begin{aligned}
\int_d^\infty \frac{(x-d)x^2e^{-\frac{x}{400}}}{2\times 400^3}\,dx &= 1200\int_d^\infty \frac{x^3e^{-\frac{x}{400}}}{6\times 400^4}\,dx - d\int_d^\infty \frac{x^2e^{-\frac{x}{400}}}{2\times 400^3}\,dx\\
&= \frac{1200}{6\times 400^4}\left(\left[-400x^3e^{-\frac{x}{400}}\right]_d^\infty + \int_d^\infty 1200x^2e^{-\frac{x}{400}}\,dx\right) - d\int_d^\infty \frac{x^2e^{-\frac{x}{400}}}{2\times 400^3}\,dx\\
&= \frac{1}{2\times 400^3}400d^3e^{-\frac{d}{400}} + \frac{1}{2\times 400^3}(1200-d)\int_d^\infty x^2e^{-\frac{x}{400}}\,dx\\
&= \frac{d^3e^{-\frac{d}{400}}}{2\times 400^2} + \frac{1200-d}{2\times 400^3}\left(\left[-400x^2e^{-\frac{x}{400}}\right]_d^\infty + 400\int_d^\infty 2xe^{-\frac{x}{400}}\,dx\right)\\
&= \frac{d^3e^{-\frac{d}{400}}}{2\times 400^2} + \frac{1200-d}{2\times 400^3}\left(400d^2e^{-\frac{d}{400}} + 400\int_d^\infty 2xe^{-\frac{x}{400}}\,dx\right)\\
&= \frac{d^3e^{-\frac{d}{400}}}{2\times 400^2} + \frac{1200-d}{2\times 400^3}\left(400d^2e^{-\frac{d}{400}} + 800\left(\left[-400xe^{-\frac{x}{400}}\right]_d^\infty + 400\int_d^\infty e^{-\frac{x}{400}}\,dx\right)\right)\\
&= \frac{d^3e^{-\frac{d}{400}}}{2\times 400^2} + \frac{1200-d}{2\times 400^3}\left(400d^2e^{-\frac{d}{400}} + 800\left(400de^{-\frac{d}{400}} + 400^2e^{-\frac{d}{400}}\right)\right)\\
&= e^{-\frac{d}{400}}\left(\frac{d^3}{2\times 400^2} + (1200-d)\left(\frac{d^2}{2\times 400^2} + \frac{d}{400} + 1\right)\right)\\
&= e^{-\frac{d}{400}}\left(1200 + 2d + \frac{d^2}{800}\right)
\end{aligned}$$

The probability that a loss results in a claim is

$$\int_d^\infty \frac{x^2e^{-\frac{x}{400}}}{2\times 400^3}\,dx = e^{-\frac{d}{400}}\left(\frac{d^2}{2\times 400^2}+\frac{d}{400}+1\right)$$

Thus, the expected payment per claim is $\frac{1200+2d+\frac{d^2}{800}}{\frac{d^2}{2\times 400^2}+\frac{d}{400}+1} = 400\frac{1200+2d+\frac{d^2}{800}}{\frac{d^2}{800}+d+400} = 400\frac{d^2+1600d+960000}{d^2+800d+320000}$

2. *The severity of a loss on a fire insurance policy follows a Pareto distribution with $\alpha = 1.4$ and $\theta = 4000$. Calculate the loss eliminatrion ratio of a deductible of $5,000.*

   Without the deductible, the expected payment per loss is $\frac{4000}{0.4} = 10000$. With the deductible, the expected payment is

$$\begin{aligned}\int_d^\infty \left(\frac{4000}{4000+x}\right)^{1.4} dx &= 4000^{1.4}\int_{d+4000}^\infty u^{-1.4}\,du \\ &= 4000^{1.4}\left[-2.5u^{-0.4}\right]_{d+4000}^\infty \\ &= 2.5\times\frac{4000^{1.4}}{(4000+d)^{0.4}} \\ &= 2.5\times\frac{4000^{1.4}}{9000^{0.4}} \\ &= 7229.811808\end{aligned}$$

   Therefore the loss elimination ratio is

$$1-\frac{7229.811808}{10000} = 27.70\%$$

3. *An insurance company has a policy where losses follow a log-logistic distribution with $\tau = 0.5$ and $\theta = 6000$. The company wants the TVaR at the 95% level for this policy to be $3,000,000. What policy limit should the company put on the policy to achieve this?*

   (i) $3,076,044

   (ii) $3,140,336

   (iii) $3,622,541

   (iv) $4,102,421

   Justify your answer.

   The survival function of the log-logistic distribition is $S(x) = \frac{\sqrt{6000}}{\sqrt{6000}+\sqrt{x}}$.

The VaR at the 95% level is therefore obtained by solving

$$\begin{aligned}
\frac{\sqrt{6000}}{\sqrt{6000}+\sqrt{x}} &= 0.05 \\
\sqrt{x}+\sqrt{6000} &= 20\sqrt{6000} \\
x &= 361 \times 6000 \\
x &= 2166000
\end{aligned}$$

With limit u, the TVaR is

$$\begin{aligned}
\mathrm{TVaR}_{0.95}(X) &= 2166000 + 20\int_{2166000}^{u} S(x)\,dx \\
&= 2166000 + 20\int_{2166000}^{u} \frac{\sqrt{6000}}{\sqrt{6000}+\sqrt{x}}\,dx \\
&= 2166000 + 20\int_{\sqrt{2166000}+\sqrt{6000}}^{\sqrt{u}+\sqrt{6000}} \frac{\sqrt{6000}}{v} 2(v-\sqrt{6000})\,dv \\
&= 2166000 + 40\sqrt{6000}\int_{\sqrt{2166000}+\sqrt{6000}}^{\sqrt{u}+\sqrt{6000}} 1-\frac{\sqrt{6000}}{v}\,dv \\
&= 2166000 + 40\sqrt{6000}\left(\sqrt{u}-\sqrt{2166000}\right) - 40\times 6000\left(\log(\sqrt{u}+\sqrt{6000}) - \log(\sqrt{2166000} -\right. \\
&= 724295.206546 + 40\sqrt{6000}\sqrt{u} - 40\times 6000\left(\log(\sqrt{u}+\sqrt{6000})\right)
\end{aligned}$$

where we have used the substitution $v = \sqrt{6000}+\sqrt{x}$ with $x = (v - \sqrt{6000})^2$, so $\frac{dx}{dv} = 2(v-\sqrt{6000})$. We therefore need to solve

$$= 724001.38696 + 293.819585906 + 40\sqrt{6000}\sqrt{u} - 40\times 6000\left(\log(\sqrt{u}+\sqrt{6000})\right) \quad = 3000000$$

We try the values given:

u	$\mathrm{TVaR}_{0.95}(X)$
(i) 3076044	3000000
(ii) 3140336	3054118
(iii) 3622541	3444173
(iv) 4102421	3808267

So (i) $u = \$3,076,044$ is the policy limit that achieves this TVaR.

4. *Aggregate payments have a compound distribution. The frequency distribution is negative binomial with $r = 5.1$ and $\beta = 0.2$. The severity distribution has mean 3,940 and variance 25,145,000. Use a Pareto approximation to aggregate payments to estimate the expected payment on a reinsurance policy with attachment point \$100,000.*

The frequency distribution has mean $5.1 \times 0.2 = 1.02$ and variance $5.1 \times 0.2 \times 1.2 = 1.224$. Therefore the aggregate loss distribution has mean $1.02 \times 3940 = 4018.8$ and variance $1.02 \times 25145000 + 1.224 \times 3940^2 = 44648786.4$. Setting these equal to the mean and variance of a Pareto distribution with parameters α and θ gives

$$\begin{aligned}
\frac{\theta}{\alpha-1} &= 4018.8 \\
\frac{\alpha\theta}{(\alpha-1)^2(\alpha-2)} &= 44648786.4 \\
\frac{\alpha}{\alpha-2} &= \frac{44648786.4}{4018.8^2} = 2.76450176556 \\
1-\frac{2}{\alpha} &= 0.361728833911 \\
\alpha &= 3.13346443684 \\
\theta &= 4018.8 \times 2.13346443684 = 8573.96687877
\end{aligned}$$

For these parameters, the expected payment on a reinsurance policy with attachment point \$100,000 is

$$\begin{aligned}
\int_{100000}^{\infty}\left(\frac{8573.96687877}{8573.96687877+x}\right)^{3.13346443684} dx &= 8573.96687877^{3.13346443684}\int_{108573.96687877}^{\infty} u^{-3.13346443684}\,du \\
&= 8573.96687877^{3.13346443684}\left[-\frac{u^{-2.13346443684}}{2.13346443684}\right]_{108573.96687877}^{\infty} \\
&= \frac{8573.96687877^{3.13346443684}}{2.13346443684 \times 108573.96687877^{2.13346443684}} \\
&= \$17.86
\end{aligned}$$

Standard Questions

5. *For a certain insurance policy, losses follow a Weibull distribution with $\tau = 2$ and $\theta = 1,000$. The policy limit of \$2,000 is applied after the deductible. The deductible is set to achieve a loss elimination ratio of 15%. What deductible achieves this loss elimination ratio?*

   (i) \$88

   (ii) \$135

   (iii) \$194

   (iv) \$284

Justify your answer

Without the deductible, the expected payment per loss is

$$\begin{aligned}\int_0^u S(x)\,dx &= \int_0^u e^{-\left(\frac{x}{1000}\right)^2}\,dx\\ &= \int_0^u e^{-\left(\frac{x}{1000}\right)^2}\,dx\\ &= 1000\sqrt{\pi}\left(\Phi\left(\frac{u\sqrt{2}}{1000}\right)-\frac{1}{2}\right)\end{aligned}$$

The expected payment after introducing a deductible d is

$$\begin{aligned}\int_d^{d+u} S(x)\,dx &= \int_d^{d+u} e^{-\left(\frac{x}{1000}\right)^2}\,dx\\ &= 1000\sqrt{\pi}\left(\Phi\left(\frac{(d+u)\sqrt{2}}{1000}\right)-\Phi\left(\frac{d\sqrt{2}}{1000}\right)\right)\end{aligned}$$

We therefore want to set

$$\begin{aligned}\frac{\Phi\left(\frac{(d+u)\sqrt{2}}{1000}\right)-\Phi\left(\frac{d\sqrt{2}}{1000}\right)}{\Phi\left(\frac{u\sqrt{2}}{1000}\right)-\frac{1}{2}} &= 0.85\\ \Phi\left(\frac{d\sqrt{2}}{1000}+2\sqrt{2}\right)-\Phi\left(\frac{d\sqrt{2}}{1000}\right) &= 0.85\left(\Phi\left(2\sqrt{2}\right)-\frac{1}{2}\right)\\ \Phi\left(\frac{d\sqrt{2}}{1000}+2\sqrt{2}\right)-\Phi\left(\frac{d\sqrt{2}}{1000}\right) &= 0.4230120\end{aligned}$$

We try the given values of d to see which one works:

d	$\Phi\left(\frac{d\sqrt{2}}{1000}+2\sqrt{2}\right)$	$\Phi\left(\frac{d\sqrt{2}}{1000}+2\sqrt{2}\right)$	$\Phi\left(\frac{d\sqrt{2}}{1000}+2\sqrt{2}\right)-\Phi\left(\frac{d\sqrt{2}}{1000}\right)$
(i) \$88	0.9984259	0.5495208	0.4489050
(ii) \$135	0.9987334	0.5757054	0.4230280
(iii) \$194	0.9990415	0.6080950	0.3909464
(iv) \$284	0.9993812	0.6560243	0.3433569

So (ii) $d = 135$ achieves the desired loss elimination ratio.

6. *An insurance company models loss frequency as negative binomial with $r = 0.1$ and $\beta = 360$, and loss severity as inverse Pareto with $\alpha = 3$, and $\theta = 1500$. The insurer sets a policy limit $u = \$30,000$ per loss. The*

insurer buys stop-loss reinsurance for aggregate losses above 1.2 times the expected aggregate losses, the price for which is based on using a Pareto distribution for aggregate losses with parameters fitted using the method of moments. The insurer's loading is 25% for the whole policy, including the ceded part, and the insurer pays 45% of its total premiums to the reinsurer. What is the loading on the reinsurance policy?

The negative binomial distribution has mean $0.1 \times 360 = 36$ and variance $0.1 \times 360 \times 361 = 12996$. The expected payment on the inverse Pareto distribution with policy limit $u = 20\theta$ is

$$\begin{aligned}
\theta \int_0^{20} 1 - \frac{x^3}{(1+x)^2}\,dx &= 1500 \int_1^{21} 1 - (u-1)^3 u^{-3}\,du \\
&= 1500 \int_1^{21} 3u^{-1} - 3u^{-2} + u^{-3}\,du \\
&= 1500 \left(3\log(21) - 3 + \frac{3}{21} + \frac{1}{2} - \frac{1}{2 \times 21^2}\right) \\
&= 10162.9360038
\end{aligned}$$

The expected square of the payment with policy limit $u = 20\theta$ is

$$\begin{aligned}
\theta^2 \int_0^{20} 2xS(x)\,dx &= 1500^2 \int_0^{20} 2x\left(1 - x^3(1+x)^{-3}\right)\,dx \\
&= 1500^2 \int_1^{21} 2(u-1)\left(1 - (u-1)^3 u^{-3}\right)\,dx \\
&= 2 \times 1500^2 \int_1^{21} (u-1)\left(3u^{-1} - 3u^{-2} + u^{-3}\right)\,dx \\
&= 2 \times 1500^2 \int_1^{21} 3 - 6u^{-1} + 4u^{-2} - u^{-3}\,dx \\
&= 2 \times 1500^2 \left[3u - 6\log(u) - 4u^{-1} + \frac{u^{-2}}{2}\right]_1^{21} \\
&= 2 \times 1500^2 \left(60 - 6\log(21) + 4 - \frac{4}{21} - \frac{1}{2} + \frac{1}{2 \times 21^2}\right) \\
&= 202695853.366
\end{aligned}$$

The variance of the loss is therefore $202695853.366 - 10162.9360038^2 = 99410585.149$ The aggregate loss therefore has mean $36 \times 10162.9360038 = 365865.696137$ and variance $36 \times 99410585.149 + 12996 \times 10162.9360038^2 = 1345874126820$. For the Pareto approximation to aggregate losses, the parameters are given by solving

$$
\begin{aligned}
\frac{\theta}{\alpha-1} &= 365865.696137 \\
\frac{\theta^2\alpha}{(\alpha-1)^2(\alpha-2)} &= 1345874126820 \\
\frac{(\alpha-2)}{\alpha} &= \frac{365865.696137^2}{1345874126820} = 0.0994578207148 \\
\alpha &= \frac{2}{1-0.0994578207148} = 2.22088431392 \\
\theta &= 365865.696137 \times 1.22088431392 = 446679.689415
\end{aligned}
$$

Under this model, the expected payment on the reinsurance policy is

$$
\begin{aligned}
\int_a^\infty \left(\frac{\theta}{\theta+x}\right)^\alpha dx &= \int_{\theta+a}^\infty \theta^\alpha u^{-\alpha}\,du \\
&= \frac{\theta^\alpha}{\alpha-1}\left[-u^{1-\alpha}\right]_{\theta+a}^\infty \\
&= \frac{\theta^\alpha}{\alpha-1}(\theta+a)^{1-\alpha}
\end{aligned}
$$

We have $a = 1.2\frac{\theta}{\alpha-1}$, so the expected payment on the reinsurance is

$$
\begin{aligned}
\frac{\theta^\alpha}{\alpha-1}(\theta+a)^{1-\alpha} &= \frac{\theta^\alpha}{\alpha-1}\left(\theta\left(1+\frac{1.2}{\alpha-1}\right)\right)^{1-\alpha} \\
&= \theta\frac{\left(1+\frac{1.2}{\alpha-1}\right)^{1-\alpha}}{\alpha-1} \\
&= 446679.689415\frac{\left(1+\frac{1.2}{1.22088431392}\right)^{-1.22088431392}}{1.22088431392} \\
&= 158618.328123
\end{aligned}
$$

The total premiums are $1.25 \times 365865.696137 = 457332.120171$, so the premium for the reinsurance is $0.45 \times 457332.120171 = 205799.454077$. The loading on the reinsurance is therefore $\frac{205799.454077}{158618.328123} - 1 = 29.745\%$.