ACSC/STAT 3703, Actuarial Models I

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Homework Sheet 8

Model Solutions

1. *An insurance company has the following portfolio of inland marine insurance policies:*

Type of policy	*Number*	*Probability of claim*	*mean claim*	*standard deviation*
Trucking	*570*	*0.0833*	*$1,502*	*$4,820*
Rail	*490*	*0.0142*	*$14,814*	*$89,241*
Other	*220*	*0.0351*	*$4,424*	*$6,395*

They model aggregate losses using a Pareto distribution. Calculate the cost of reinsuring losses above $2,000,000, if there is a 35% loading on the reinsurance premium.

We calculate the mean and variance of the aggregate loss:

Type of policy	$\mathbb{E}(N)$	$\mathrm{Var}(N)$	mean aggregate loss	var aggregate loss
Trucking	47.481	43.5258327	71316.462	1201292037.07
Rail	6.958	6.8591964	103075.812	56918488586.5
Other	7.722	7.4509578	34162.128	461627582.097
Total			208554.402	58581408205.7

Using a Pareto approximation, the method of moments gives the following parameters

$$\begin{aligned}
\frac{\theta}{\alpha-1} &= 208554.402\\
\frac{\alpha\theta^2}{(\alpha-1)^2(\alpha-2)} &= 58581408205.7\\
\frac{\alpha}{\alpha-2} &= \frac{58581408205.7}{208554.402^2} = 1.34685575149\\
1-\frac{2}{\alpha} &= 0.742470007567\\
\alpha &= 7.76608573279\\
\theta &= 1411096.96388
\end{aligned}$$

The expected reinsurance payment is therefore

$$
\begin{aligned}
\int_{2000000}^{\infty} \left(\frac{\theta}{\theta+x}\right)^{\alpha} dx &= \int_{2000000+\theta}^{\infty} \theta^{\alpha} u^{-\alpha}\, du \\
&= \frac{1411096.96388^{7.76608573279}(3411096.96388)^{-6.76608573279}}{6.76608573279} \\
&= 531.528983709
\end{aligned}
$$

With a 35% loading The premium is therefore $531.528983709 \times 1.35 =$ \$717.56.

2. *An insurance company sells Workers' Compensation insurance. It estimates that the standard deviation of the aggregate annual claim is \$290,000 and the mean is \$26,000.*

   (a) How many years history are needed for a company to be assigned full credibility? (Use $r = 0.1$, $p = 0.95$.)

   The coefficient of variation for aggregate annual claim is $\frac{290000}{26000} = \frac{145}{13}$. For the average of n years of aggregate claims, the coefficient of variation is $\frac{145}{13\sqrt{n}}$. Using $r = 0.1$ and $p = 0.95$, the standard for full credibility is obtained by solving:

$$
\begin{aligned}
P\left(\left|\frac{\overline{X}-\mu}{\mu}\right| < 0.1\right) &> 0.95 \\
2\Phi\left(\frac{0.1\times 13\sqrt{n}}{145}\right) - 1 &> 0.95 \\
\frac{1.3\sqrt{n}}{145} &> 1.96 \\
n &> \left(\frac{145\times 1.96}{1.3}\right)^2 \\
&= 47792.6863904
\end{aligned}
$$

   so 47793 years are needed.

   The standard net premium for this policy is \$26000. A company has claimed a total of \$136,902 in the last 18 years.

   (b) What is the net Credibility premium for this company, using limited fluctuation credibility?

   The credibility of 18 years of experience is $Z = \sqrt{\frac{18}{47792.6863904}} = 0.0194068715456$. The premium for this company is therefore $0.0194068715456 \times \frac{136902}{18} + 0.980593128454 \times 26000 =$ \$25643.02.

Standard Questions

3. *A home insurer divides policyholders into three categories: Urban; Suburban; and Rural. The number of claims made by a policyholder follows a Poisson distribution with a certain mean λ, depending on the type of home. The insurance company has the following portfolio of policies.*

Category	*Number insured*	λ	*mean claim*	*standard deviation of claim*
Urban	*2884*	*0.05*	*1309*	*52401*
Suburban	N	*0.02*	*1624*	*93228*
Rural	*844*	*0.09*	*4014*	*169212*

The insurance company is currently advertising heavily in suburban areas and hopes that this will increase the number of policies sold to those homeowners. The insurer models aggregate losses as following a Pareto distribution. The insurer charges a loading of 20% of the expected payments. It buys stop-loss reinsurance with attachment point 1.1 times expected payments. The reinsurer uses the same Pareto distribution to model aggregate losses, and charges a loading of 40% for the reinsurance. The insurer wants the total reinsurance payment to be less than 10% of premiums received. How many additional suburban policies does it need to sell in order to achieve this?

If the number of claims for each policyholder follows a Poisson distribution, then the number of aggregate claims for each class also follows a Poisson distribution. The mean and variance of the aggregate claims are both proportional to λ. Let N be the number of claims sold to suburban homeowners

Category	λ	$\frac{\mathbb{E}(S)}{\lambda}$	$\mathbb{E}(S)$	$\frac{\mathrm{Var}(S)}{\lambda}$	$\mathrm{Var}(S)$
Urban	144.2	1309	188757.80	2747578282	396200788264
Suburban	$0.02N$	1624	$32.48N$	8694097360	$173881947.2N$
Rural	75.96	4014	304903.44	28648813140	2176163846110
Total			$32.48N + 493661.24$		$173881947.2N + 2572364634374$

Setting the aggregate mean and variance as the mean and variance of a Pareto distribution gives

$$\frac{\theta}{\alpha-1} = 32.48N + 493661.24$$

$$\frac{\alpha\theta}{(\alpha-1)^2(\alpha-2)} = 173881947.2N + 2572364634374$$

$$\frac{\alpha-2}{\alpha} = \frac{(32.48N + 493661.24)^2}{173881947.2N + 2572364634374}$$

For reinsurance with attachment point $a = 1.1\frac{\theta}{\alpha-1}$, the expected payment is

$$\begin{aligned}
\int_a^\infty \left(\frac{\theta}{\theta+x}\right)^\alpha dx &= \int_{a+\theta}^\infty \theta^\alpha u^{-\alpha}\,dx \\
&= \theta^\alpha \left[\frac{u^{1-\alpha}}{(1-\alpha)}\right]_{a+\theta}^\infty \\
&= \frac{\theta^\alpha}{(\alpha-1)(a+\theta)^{\alpha-1}} \\
&= \frac{\theta^\alpha}{(\alpha-1)(1.1\frac{\theta}{\alpha-1}+\theta)^{\alpha-1}} \\
&= \frac{\theta^\alpha(\alpha-1)^{\alpha-2}}{((\alpha+0.1)\theta)^{\alpha-1}} \\
&= \frac{\theta(\alpha-1)^{\alpha-2}}{(\alpha+0.1)^{\alpha-1}}
\end{aligned}$$

The total expected payment on the insurance is $\frac{\theta}{\alpha-1}$, so the insurer's premium is $1.2\frac{\theta}{\alpha-1}$, and the reinsurance premium is $1.4\frac{\theta(\alpha-1)^{\alpha-2}}{(\alpha+0.1)^{\alpha-1}}$. Thus, the insurer wants to ensure that $1.4\frac{\theta(\alpha-1)^{\alpha-2}}{(\alpha+0.1)^{\alpha-1}} \leqslant 0.12\frac{\theta}{\alpha-1}$ or $(\alpha+0.1)^{\alpha-1} \geqslant 11.6666666667(\alpha-1)^{\alpha-1}$ Numerically, we solve this to get $\alpha \geqslant 3.606500$. [There is another solution for $\alpha \leqslant 1.087002$, but 3.6065 is the solution we need.]

Therefore, in order to achieve the required cost for reinsurance, the insurer needs $\alpha \geqslant 3.606500$ or $\frac{\alpha-2}{\alpha} \leqslant 0.445445723$. From the calculations above, this means the number N of suburban policies sold needs to satisfy

$$\begin{aligned}
\frac{(32.48N+493661.24)^2}{173881947.2N+2572364634374} &\leqslant 0.445445723 \\
(32.48N+493661.24)^2 &\leqslant 0.445445723(173881947.2N+2572364634374) \\
1054.9504N^2+32068234.1504N+243701419878 &\leqslant 77454969.6872N+1.145848824380 \\
1054.9504N^2-45386735.5368N-902147404502 &\leqslant 0 \\
N &\in \frac{45386735.5368 \pm \sqrt{45386735.5368^2+4\times 1054.9504 \times}}{2\times 1054.9504} \\
&= [-14791.4806711, 57814.1057507]
\end{aligned}$$

Thus, they need to ensure $N < 57814.1057507$.

4. *An auto insurance company sets the standard for full credibility as 1264 car-years. The book estimates are 0.17 claims per car-year for claim frequency and $1,885 per claim for claim severity.*

   The company changes the standard to 820 car-years for frequency and 232 claims for severity. For a particular policyholder with 17 car-years of experience, who claimed a total of $20,975.07 in that time, this results in a 2% decrease in premiums. How many claims did this policyholder make?

   Since the policyholder claimed $20,975.07 in 17 car-years, their average aggregate claim per car-year was $\frac{20975.07}{17} = 1233.82764706$. Using the original standard, the credibility of 17 car-years was $Z = \sqrt{\frac{17}{1264}} = 0.115971406341$, so the policyholders premium was $0.115971406341 \times 1233.82764706 + 0.884028593659 \times 0.17 \times 1885 = \$426.37569025$ A 2% decrease means that under the new method, the policyholders premium is $0.98 \times 426.37569025 = 417.848176445$.

   Let the number of claims be n. Under the new method, the credibility of 17 car-years for frequency is $Z = \sqrt{\frac{17}{820}} = 0.14398509408$ so this policyholders estimated claim frequency is

   $$0.14398509408\frac{n}{17} + 0.85601490592 \times 0.17 = 0.00846971141647n + 0.145522534006$$

   The credibility of n claims for estimating severity is $Z = \sqrt{\frac{n}{232}}$, so this policyholders estimated severity is

   $$\sqrt{\frac{n}{232}}\frac{20975.07}{n} + 1885\left(1 - \sqrt{\frac{n}{232}}\right) = \frac{1377.08081034}{\sqrt{n}} + 1885 - 123.756312971\sqrt{n}$$

   . The policyholder's new premium is therefore

   $$(0.00846971141647n + 0.145522534006)\left(\frac{1377.08081034}{\sqrt{n}} + 1885 - 123.756312971\sqrt{n}\right)$$

   We need to find the value of n which makes this equal to 417.848176445. That is, we need to solve (letting $x = \sqrt{n}$)

   $$(0.00846971141647n + 0.145522534006)\left(\frac{1377.08081034}{\sqrt{n}} + 1885 - 123.756312971\sqrt{n}\right) = 417.8481764452$$

   $$(0.00846971141647x^2 + 0.145522534006)\left(81.0049080278 + 1885x - 123.756312971x^2\right) = 417.848176445x$$

   We solve this numerically to get $x = 2.2360$ or $x = 1.9281$, so $n = 5$ or $n = 3.71756961$. Since the number of claims must be an integer, we get $n = 5$.