ACSC/STAT 3703, Actuarial Models I
Further Probability with Applications to Actuarial Science
Winter 2025
Toby Kenney
In Class Examples

Outline

Outline

Outline

Outline

Outline

Outline

Outline

Outline

Outline

Outline

Outline

Outline

Outline

Utility functions

An individual's (or institution's) utility function represents how much happiness is derived from a certain amount of wealth.

The idea is that wealth is first used to buy more important things, and as wealth increases, the new things that can be bought are less important, so contribute a smaller increase in happiness.

Generally, wealth is uncertain, and people and institutions aim to maximise expected utility.

Properties of Utility Functions

- Utility functions are increasing — more wealth is good.
- Utility functions are usually concave — risk averse.

Question 1

An individual has utility function $u(x) = 7000x - x^2$ for $0 \leqslant x < 5000$, and has current wealth \$4,000. This individual faces a 5% risk of losing \$2,000. Insurance is available for a premium of \$120. Should the individual purchase insurance?

What Risks are Insurable

- Economically feasible — Concavity of Utility exceeds expenses.
- Risk is estimable — enough available data.
- Loss is well-defined — avoid manipulation & moral hazard.
- Loss is random.
- Exposures are homogeneous — this may not hold due to negative selection.
- Losses from different policyholders are independent.

IRLRPCI 2.2 Automobile insurance

Types of Automobile-insurance coverage

- Liability insurance
- Medical Benefits
- Uninsured and underinsured motorist coverage
- Collision insurance
- Other than collision (OTC) insurance

Notes

- First two coverages often legally required.
- Other coverages may be required in order to use car as security for a loan.
- Policy usually covers policyholder and immediate family on listed vehicles. May also cover invited drivers.
- Covers listed vehicles and usually also attached trailers, etc.
- Other policies apply to commercial vehicles.

IRLRPCI 2.2 Automobile insurance

Tort system

In this system, fault for an accident is legally determined through a court case, or settlement. At-fault party is legally responsible for all costs.

No-fault system

Injured party is covered through their own insurance, with no need to determine fault for the accident. Many different details for how this system can work — e.g. threshold no-fault or government monopoly. Even in this system, determination of fault may be performed to determine future premiums.

Question 2

What are the advantages and disadvantages of each system?

Answer to Question 2

Advantages of tort system	Advantages of no-fault system
Increases coverage costs for at-fault drivers, thus increasing the incentive to drive carefully.	Reduces litigation costs
More flexibility to taylor payments to injured parties needs. Under no-fault system, benefits usually defined by a formula.	Evidence shows that under tort system, small claims are over-compensated, whereas larger claims are undercompensated.

IRLRPCI 2.2 Automobile insurance

Liability insurance

- Covers costs of policyholder's damage to third parties.
- Can cover legal costs of policyholder.
- Policy limits usually apply to damage payments, not legal costs.
- Insurer may however stop paying legal costs once its damage payments have already exceeded the policy limit.
- This insurance is compulsory almost everywhere, with minimum legal limits. It may be advisable to buy increased policy limits if these are low.
- Premiums vary with policy limits, location, use of automobile, driving record of policyholder, age, sex and marital status of the policyholder.

Medical benefits

- Sometimes called medical payments, personal injury protection or accident benefits.
- Covers costs (e.g. medical costs, income replacement, survivors benefits, rehabilitation costs, home care costs) arising from injury to policyholder (unless another party is liable).
- Usually compulsory
- May not apply to commercial vehicles where these benefits are covered by workers compensation.
- In tort juristictions, liability insurance is more important (and accounts for more of the costs). In no-fault juristictions, medical benefit insurance is more important, and accounts for more costs.

Uninsured and underinsured motorist coverage

- Covers costs to policyholder if injured by:
  - an unidentified driver
  - an uninsured driver
  - an underinsured driver (if liable driver's coverage is lower than policyholder's)

Collision insurance and OTC insurance

- Cover damage to policyholder's vehicle.
- loss is defined as lesser of cash value of damaged property or cost to repair/replace. May include special provisions for cases where cash value (with depreciation) exceeds outstanding balance on a loan secured by the property.
- Usually include a deductible.
- In a tort juristiction, the insurer who pays the benefits can sue the at-fault driver. If the suit is successful, the insurer will reimburse the deductible to the policyholder, and use the rest to cover the payment. This is known as subrogation.
- Subrogation reduces the cost of collision insurance, and increases the cost of liability insurance.
- If the insurer decides not to sue, the policyholder can sue for their deductible, and any costs exceeding the policy limit.

Collision insurance and OTC insurance (cont.)

- If vehicle is "written-off", the insurer has the right of salvage — any scrap value of vehicle belongs to insurer. If scrap value exceeds amount paid, the insurer must increase its payment accordingly.
- Premiums for collision and OTC insurance depend on:
  - Type of car (based on value of car and cost of repairs).
  - Use of car
  - Location.
  - Driver's history.
  - Where allowed by law: age, gender, marital status.
- OTC covers fire, weather, vandalism, stone chips, theft, etc.
- Usually excludes: war, terrorism, wear & tear, road damage to tyres, radioactive contamination, and collision.
- Comprehensive includes any cause not specifically excluded. Under specified perils, only a given list of causes are reimbursed.
- OTC premiums usually only vary by vehicle type and location.

Four Coverages in Homeowner's Insurance

- Coverage A — Damage to home
- Coverage B — Damage to garage or other structures
- Coverage C — Personal property in home
- Coverage D — Living expenses and loss of rental income
- Section II — Liability

Doctrine of Proximate Cause

- Coverage might be comprehensive or specified perils.
- Proximate cause is a legal definition of when one event can be considered to have caused another. The proximate cause must be directly linked to the damage by a chain of events without other independent causes.
- For a claim to be payable, an insured peril must be a proximate cause of an insured loss.

IRLRPCI 2.3 Homeowner's insurance

Coverage A

- Includes a deductible (may decrease to zero for larger losses).
- Subrogation can apply here if a third party is liable for the damage.
- Policy limits less than 80% of the value of the house (at time of loss) result in coinsurance for smaller claims.
- Increases in house prices could result in a homeowner falling below the 80% cut-off by accident.
- Many insurers offer an option for the premium to increase annually in line with a specified index, and to waive the 80% requirement.

Question 3

A homeowner's house is valued at $350,000. However, the home is insured only to a value of $260,000. The insurer requires 80% coverage for full insurance. The home sustains $70,000 of water damage due to a burst pipe. How much does the insurer reimburse? (There is no deductible for losses above $2,000.)

IRLRPCI 2.3 Homeowner's insurance

Coverage B — Garage

- Usually up to 10% value of house.
- More coverage can be purchased for extra premium.

Coverage C — Personal Property

- Limit usually 40–50% of house value
- Often inside limits on each type of item.
- For full insurance on jewellry, silverware or art, policyholder can provide a schedule showing appraised values of these items. If lost or stolen, this appraised value is paid by insurer (not the current market value).
- Extends to borrowed property in policyholder's possession.
- Also applies to personal property lost or damaged outside the home.

IRLRPCI 2.3 Homeowner's insurance

Coverage D — Accommodation and Loss of Rental Income

- Covers fair rental value for alternative accommodation while repairs are made to home.
- Also covers lost rental income from any part of the home that lost while damage is repaired.

Section II — Liability

- Liability could arise if a third party is injured or property is damaged while on the property.
- Only applies in case of negligence by homeowner.
- Most claims settled out of court.
- Insurance will pay policyholder's defence costs unless liability payments already made exceed policy limits.
- Limited medical coverage for injured third-parties on a no-fault basis.

Comments on Homeowner's Insurance

- Premiums depend on location, construction and value.
- In high-risk areas for floods and earthquakes, these perils are often excluded.
- Coverage for these excluded perils can be purchased for an extra premium.
- Construction may affect the risk of various perils.
- Discounts may be offered for security systems or sprinkler systems, etc.

IRLRPCI 2.4 Tennants package

Tennant insurance

- Contains provisions of homeowners insurance relevant to tennants
- Includes personal possessions.
- Generally lower liability provisions for apartments because majority of liability arises from incidents on surrounding ground, covered by landlord's insurance.
- Different policies also available for condominium owners.

Workers' Compensation

- Early type of no-fault insurance.
- Prior to 1895, getting compensation was difficult for employees.
- Objectives of Worker's Compensation:
  - Broad coverage of occupational illness and injury.
  - Protection against loss of income
  - Provide medical care and rehabilitation expenses
  - Encourage employers to provide safer workplaces.
  - Provide efficient and effective delivery of benefits.
- Workers' Compensation Board controlled by province.
- U.S. employers can choose private, self or state insurance.
- Employee must work in a covered occupation, and experience an accident or disease resulting from employment, while employed.
- Diseases develop slowly, generally more expensive, often subject to disagreements about extent to which they are caused by work.

Typical Benefits

- Unlimited medical care benefits
- Disability income benefits, waiting period 3–7 days, percentage of salary, depending on degree of disability. Degree of disability is usually classified as:
  - Temporary but total
  - Permanent and total
  - Temporary and partial
  - or Permanent but partial
- Death benefits
- Rehabilitation benefits and services.
- Premiums depend on salaries of employees, industry class, etc.

IRLRPCI 2.6 Fire insurance

Fire insurance

- Included in homeowner insurance and tennants insurance.
- Policies provide protection for commercial properties.
- Originally only fire was an insured peril, but many more perils now covered, even some comprehensive policies.
- Covers both direct and indirect loss.
- Standard Fire Policy covers direct loss from fire and lightning.
- At least one additional form must be added. Common forms:
  - Include personal coverage
  - Include commercial coverage
  - Increase covered perils (e.g. vandalism, malicious mischief, etc.)
  - Increase covered losses (e.g. living expenses, rental income, leasehold interests, demolition, business interruption losses)
- Allied lines are additional coverage sold in separate policies, e.g. earthquake, rain, sprinkler leakage, water damage, crop hail.
- Larger corporations may design own forms to meet specific needs.

IRLRPCI 2.7 Marine insurance

Ocean Marine Insurance

- Covers oceangoing ships and cargo. Covers shipowner's liability.
- Basic policy only covers cargo while loaded onto ship.
- Some policies provide warehouse-to-warehouse coverage.

Inland marine insurance

- A modification of marine insurance for the trucking industry.
- Covers transportation of goods by railway, motor vehicle, ship or barge on inland waterways (e.g. canals and rivers) or coastal trade, air, mail, armoured car or messenger.
- Also covers infrastructure for transportation — bridges, tunnels, wharves, docks, communication equipment, moveable property.
- May have additional coverage for construction equipment, personal jewellry and furs, agricultural equipment, and animals.

IRLRPCI 2.8 Liability insurance

Types of Liability Insurance

- Included by default in auto and homeowner's insurance.
- Product liability insurance
- Errors and omissions Insurance
- Medical malpractice insurance
- Professional liability insurance

Features of Liability Insurance

- Low frequency high value claims.
- Claims often reported years after event.
- Claims often take many more years to settle after reporting.
- Claims-made policy form covers only claims after a specified date reported during policy period.
- Tail coverage sold to cover claims reported after the period.
- High litigation cost. Sometimes policy limit applies to legal costs.

IRLRPCI 2.9 Limits to Coverage

Deductibles

- Reduce claim amount by a small amount from loss amount.
- Claim paid is $(X - d)_+$ $X = \text{loss amount}$, $d = \text{deductible}$.

Policy Limits

- Maximum amount payable by insurer.
- Claim paid is $X \wedge l$ $X = \text{loss amount}$, $l = \text{limit}$.

Coinsurance

- Proportional reduction in claim amount.
- Claim paid is cX $X = \text{loss amount}$, $c = \text{coinsurance proportion}$.

Combining all three modifications

- Applied in order: deductible, coinsurance, limit
- Claim paid is $(c(X - d)_+) \wedge l$ $X = \text{loss amount}$, $d = \text{deductible}$, $l = \text{limit}$ and $c = \text{coinsurance proportion}$.

Question 4

An auto insurance policy has a deductible of $1,000, a policy limit of $100,000 and co-insurance such that the policyholder pays 20% of the remaining claim. How much does the insurer pay if the loss is:

a) $600
b) $2,500
c) $101,600
d) $146,900

IRLRPCI 2.9 Limits to Coverage— Deductibles

Reasons for Using Deductibles

- Small claims involve disproportionate administrative costs.
- Premium savings.
- Moral hazard.
- Better expected utility.

Problems with Deductibles

- Public relations.
- Marketing difficulties.
- Insureds may inflate claims to recover deductible.

Types of Deductible

- Fixed dollar deductibles
- Fixed %age deductibles
- Disappearing deductibles
- Franchise deductibles
- Fixed dollar deductible per year
- Elimination period

IRLRPCI 2.9 Limits to Coverage— Policy Limits

Reasons for Policy Limits

- Clarifies insurer's oblications
- Reduces risk to insurer, allowing lower premiums.
- Allows policyholder to choose appropriate coverage.

Notes on policy limits

- Policy may have different policy limits for different parts of claim.
- May also have inner limits.
- Policy limits apply to damage payments only — they do not include administrative and legal costs.

Question 5

An insurer sells 100 identical policies. Each policy has a 50% probability of incurring a loss. If a loss is incurred, the loss follows a Pareto distribution with $\alpha = 2$ and $\theta = 10000$.
(a) What is the probability that at least one policy incurs a single loss exceeding $1,000,000?
(b) If the insurer has a policy limit of $100,000 on the policy, what is the probability that the aggregate loss exceeds $1,000,000?

What is Reinsurance?

A contract to for a larger reinsurance company to pay some of the losses on a portfolio of insurance contracts.

Types of Reinsurance

- Excess of Loss — if a single loss exceeds the attachment point reinsurer will pay the excess (up to a limit).
- Stop-loss — like excess of loss for portfolio. Attachment point and limit often defined in terms of loss ratio.
- Quota share — proportional reinsurance.
- Catastrophe cover — like stop-loss, but relating to losses from a single event (e.g. earthquake, hurricane).

Why Reinsurance?

- Insurer's exposure is too concentrated (e.g. geographically) leaving insurer vulnerable to catastrophe.
- Insurer has limited ability to absorb large losses.
- Insurer has financial difficulties.
- Insurer wants to increase stability.
- Insurer wants to exit a line of business.

Question 6

An insurance company models its aggregate losses as following an exponential distribution. It can buy stop-loss reinsurance for a loading of 100% of the expected claim. The insurance company sets its premium so that total premiums equal the mean plus one standard deviation of aggregate payments. Find the attachment point for reinsurance that minimises the insurer's premiums.

Notes on Reinsurance

- For proportional reinsurance, reinsurer pays ceding commission to cover the ceding company's expense costs. Otherwise the ceding company's expense loading would be too high.

IRLRPCI 5.5 Reinsurance

Ratemaking for Reinsurance

- Treaties on risk-exposed basis (covers losses during time period) or risk-attaching basis (covers policies written during time period).
- Premium based on ceding company's earned premium for risk-exposed treaties, paid premium for risk-attaching treaties.
- Pricing may be by exposure rating (based on industry averages) or experience rating (based on ceding company's experience).
- Simulation is important to estimate variability in losses.
- Catastrophe cover priced using long experience period and catastrophe models incorporating meteorological, seismic and engineering data, to assess the likely impact of catastrophes.
- Catastrophe cover is high risk. This is reflected by high loading for profit and contingencies.

IRLRPCI 4.2 Objectives of ratemaking

Strict Requirements

- Cover expected losses and expenses.
- Make adequate provision for contingencies.
- Encourage loss control.
- Satisfy regulators.

Desirable Objectives

- Remain reasonably stable.
- Respond to changes.
- Be easy to understand.

IRLRPCI 4.2 Objectives of ratemaking

Components needed to calculate rates

- Claim frequency distribution
- Loss distribution
- Interest rate
- Times of payments.

Sources of Uncertainty

- For life contingencies, loss is usually specified, so the main sources of uncertainty are claim frequency, interest rate and times of payment.
- For property or casualty insurance, claim frequency distribution and loss distribution are important sources of uncertainty.
- For lines of insurance where settlement can take a long time (e.g. liability insurance), time of payment and rate of interest can be a source of uncertainty.

IRLRPCI 4.3 Data for ratemaking— Three Ways to Record Data

Accident Year

- All payments are recorded under the year when the loss occurred.
- Data first becomes available on 31st December of that year.
- Data originally consists of paid loss amounts plus loss reserves.
- Data are updated as claims are settled.

Policy Year

- Payments recorded under the year when policy came into force.
- Data first available on 31st December of the following year.
- Data originally consists of paid loss amounts plus loss reserves.
- Has the advantage of being under the same policy basis.

Calendar Year

- Payments recorded under the year when the payment is made.
- Includes changes to loss reserves.

Question 7

A home insurance policyholder pays $640 annual premium on 1st October 2015. What is the earned premium from this policy in
(a) 2015
(b) 2016

IRLRPCI 4.5 The exposure unit

Exposure Unit

- Measure of how exposed to loss the policy is.
- Premium calculated as $(\text{units of exposure}) \times (\text{rate per unit})$
- Examples include car-years, house value, payroll.

Good exposure units should

- Accurately measure exposure to loss
- Be easy to determine (at time of premium calculation)
- Be unable to be manipulated
- Be easy to administer
- Be easy for policyholder to understand.
- Automatically adjust with inflation.

Question 8

An actuary is reveiwing claims data from accident year 2022 to calculate premiums for policy year 2024. She finds that the expected number of claims per unit of exposure is 0.003, and the expected claim value per claim in accident year 2022 was $26,000. Payments are subject to an annual inflation rate of 3%. What pure premium should she set for 2024?
[Assume the number of policies in force is constant, and that losses occur uniformly over the year for all policies in force.]

Question 9

An insurance company starts a new line of insurance on 1st March, 2022. An actuary finds that a premium of $644 would have achieved the desired loss ratio in 2022. Assume that policies were sold uniformly during the year, starting in March; losses are distributed in proportion to the number of policies in force; and inflation is at a constant rate of 8%. What should the premium be for premium year 2025 to achieve the desired loss ratio?

IRLRPCI 4.7 Ingredients of ratemaking

Loss-Development Factors

- Work with incurred lossed (include estimated loss reserves)
- Adjust data to reduce impact of large losses.

Trend Factors

- Adjust expected premiums to future payment periods.
- Cover inflation, changes to court rulings, technical advances, etc.
- Usually estimated using least-squares (linear or non-linear) regression.
- Regression may be applied separately to frequency and severity, or to aggregate losses.
- Actuary may choose to assign more weight to more recent data.
- Should take into account external factors.
- Trends in premium can help estimate loss ratio method.

IRLRPCI 4.7 Ingredients of ratemaking

Expenses

- Usually divided between Loss Adjustment Expenses (LAE) and other expenses
- LAE are divided into allocated (ALAE) and unallocated (ULAE).
- Sometimes separate between expenses which are proportional to gross premiums and expenses proportional to exposure.

Loading for Profit and Contingencies

- Loading can be implicit or explicit (usually a percentage of gross premium).
- Implicit approach historically calculated by underestimating investment returns.
- Competition should prevent loading being too much.

IRLRPCI 4.8 Rate Changes

Overall Rate Change

- Loss cost method: $\text{New average gross rate} = \frac{\text{New Average Loss Cost}}{1-\text{Expense Ratio}}$
- Loss ratio method: $\text{Rate Change} = \frac{\text{Expected Effective Loss Ratio}}{\text{Permissible Loss Ratio}} - 1$

Question 10

The current base premium for an insurance policy is $974. An insurer has earned premiums of $85,346,200. Total losses are $76,594,400. Calculate the new base premium for the following year without inflation to achieve a permissible loss ratio of 80%.

Question 11

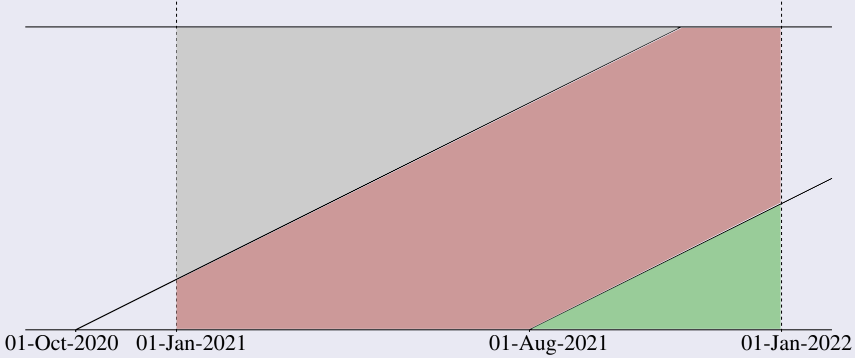
An insurer received earned premiums of $1,700,000 in 2021 for a particular line of insurance. The total loss payments were $1,520,000. There were rate changes on 1st October 2020 and on 1st August 2021. The premiums were as follows:

Before October 2017	$432
October 2017 – July 2018	$464
From August 2018	$491

Assuming that policy dates are uniformly distributed over the years, calculate a new rate for policy year 2025 to achieve a loss ratio of 80%, assuming annual inflation of 4%.

IRLRPCI 4.8 Rate Changes

Answer to Question 11



Question 12

An auto insurer had $3,679,710 in earned premiums in accident year 2021. The total losses were $3,244,610. There was a rate change on 9th July 2021 [190th day of the year], which affects some of the policies. Before the rate change, the base premium was $629. The current base premium is $660. Assuming that policies are sold uniformly over the year, calculate the new premimums for policy year 2023 assuming 6% annual inflation and a permissible loss ratio of 0.75.

What is Loss Reserving?

Estimation of the outstanding payments on a portfolio of policies.

Why is Loss Reserving Necessary?

- It sometimes takes time to determine the amount payable for a given claim.
- The insurer needs to calculate profits for each year before all claims are settled, for both corporate finance and tax reasons.

IRLRPCI 3.2 How outstanding claim payments arise

Typical Steps in a Claim Payment

1. Claim Event
2. (a) Claim reported to agent. (b) Claim recorded by insurer.
3. (a) Initial payments and settlement offer. (b) Settlement rejected.
4. Court case.

Notes

- Common for lengthy delays to occur between steps.
- Claims could remain open for 10–20 years.
- Time to settlement can vary a lot with line of insurance.
- Largest claims often settle last.

Sources of Uncertainty

- Eventual cost of claim payments for known claims.
- Claims incurred but not yet reported.

IRLRPCI 3.3 Definition of terms

Claim File

- Established as soon as field adjuster is aware of pending claim.
- Field adjuster estimates expected ultimate claim payment, and updates the estimate regularly.
- Aggregate of individual claim file estimates called case reserves.

Gross IBNR

Additional reserves above case reserves, also called bulk reserve, including provision for:

- Adjustments of case reserves
- Claim files which are closed but may reopen.
- Claims incurred but not reported (IBNR).
- Claims reported but not recorded (RBNR).

IRLRPCI 3.3 Definition of terms

Paid Loss Development

- Change in cumulative payments made between valuation dates called paid age-to-age loss development.
- Relative change called paid loss development factor. Can be less than 1 because of salvage and subrogation.
- Difference between cumulative payments made and ultimate payment amount called age-to-ultimate loss development.

Incurred Loss Development

- Incurred age-to-age loss development: change in estimated costs.
- Incurred loss-development factor is relative change.
- Incurred loss-development factors can be less than 1 if estimates were conservative or because of salvage and subrogation. Incurred loss-development factors greater than 1 indicate inadequate claim file estimates.

IRLRPCI 3.3 Definition of terms

Salvage and Subrogation

Recall that after paying a claim, insurance company acquires rights of salvage and subrogation (provided the money returned does not exceed the claim amount). These can reduce the incurred losses.

Loss adjustment expenses

- Expenses involved with settling claims — e.g. legal costs.
- Allocated loss adjustment expenses (ALAE) relate to specific claims. These become part of total claim cost.
- Unallocated loss adjustment expenses (ALAE) (e.g. rent) are allocated among calculated reserves following a formula.
- Classification can vary between insurers.

Fast Track Reserves

High frequency, low severity lines of insurance use fast track average reserves based on recent experience and trends for new claim files.

Setting loss reserves requires detailed knowledge

- Company's business. For example any changes in
  - Portfolio composition.
  - Claim administration.
  - Management.
  - Reinsurance.
- External factors. For example
  - Inflation.
  - Legal rulings.
- Format and definitions of data used by company. For example, how are claims separated into claim files? Actuary should separate data into homogeneous categories. This may involve splitting separate parts of individual claims.
- The actuary must review accuracy of data and compare multiple methods for estimating reserves. Where methods give conflicting answers, actuary must explain differences.

IRLRPCI 3.5 Checking the data

Checking for Inconsistencies.

- It is important to check for consistency of patterns accross the data.
- Where inconsistencies arise in the data, the actuary must identify the source of the inconsistency.
- Reserving actuary must document the findings and any ensuing adjustments or subjective changes to the calculations.

IRLRPCI 3.6 Loss reserving methods

Expected Loss Ratio Method

1. Calculate the expected ultimate loss ratio (ultimate claim payments divided by total earned premiums).
2. Multiply be earned premiums for period.
3. Subtract payments made to date.

Problems with this approach

- Danger of manipulation of expected loss ratio.
- Expected loss ratio will not apply after changes of premium.
- Expected loss ratio might change if portfolio changes.

Why is the method used?

- For new lines of business without past data, it may be the only available method.
- Can be used as a backup check for other methods.

Question 13

An insurance company has three types of claims with different expected loss ratios as shown in the following table:

Claim Type	Policy Year	Earned Premiums	Expected Loss Ratio	Losses paid to date
Collision	2014	$200,000	0.79	$130,000
	2015	$250,000	0.79	$110,000
	2016	$270,000	0.77	$60,000
Comprehensive	2014	$50,000	0.74	$36,600
	2015	$60,000	0.72	$44,300
	2016	$65,000	0.75	$41,400
Bodily Injury	2014	$300,000	0.73	$86,000
	2015	$500,000	0.73	$85,000
	2016	$600,000	0.72	$12,000

Use the expected loss ratio method to estimate the loss reserves needed.

Question 14

The following table shows the paid losses on claims from one line of business of an insurance company over the past 6 years.

	Development year					
Accident year	0	1	2	3	4	5
2011	5,826	659	2,910	845	1,349	120
2012	7,327	2,896	1,540	963	−348	
2013	8,302	1,719	1,380	2,031		
2014	8,849	1,701	673			
2015	9,950	320				
2016	11,290					

Assume that all payments on claims arising from accidents in 2011 have now been settled. Estimate the future payments arising each year from open claims arising from accidents in each calendar year. (That is, fill in the empty cells in the table.)

Answer to Question 14

We first construct the cumulative loss payments in the following table:

Accident year	Development year 0	1	2	3	4	5
2011	5,826	6,485	9,395	10,240	11,589	11,709
2012	7,327	10,223	11,763	12,726	12,378	
2013	8,302	10,021	11,401	13,432		
2014	8,849	10,550	11,223			
2015	9,950	10,270				
2016	11,290					

Answer to Question 14

The corresponding loss development factors are

Accident year	Development year 1/0	2/1	3/2	4/3	5/4
2011	1.113	1.449	1.090	1.132	1.010
2012	1.395	1.151	1.082	0.973	
2013	1.207	1.138	1.178		
2014	1.192	1.064			
2015	1.032				

Answer to Question 14

The corresponding loss development factors are

Accident year	Development year 1/0	2/1	3/2	4/3	5/4
2011	1.113	1.449	1.090	1.132	1.010
2012	1.395	1.151	1.082	0.973	
2013	1.207	1.138	1.178		
2014	1.192	1.064			
2015	1.032				

IRLRPCI 3.6 Loss reserving methods— Chain ladder or loss development triangle method

Approaches

- Average
- 5-year average
- Mean — weighted by claim payment amounts.

Pros and Cons of Incurred Loss Triangles

- Incurred loss estimates represent the company's best estimate of losses, including information not reflected in paid loss data.
- Paid loss data are objective, incurred loss data are subjective.
- Incurred loss estimates react immediately to changes.

Problems with chain ladder approach

- Too many parameters for the data.
- Unstable — changes in methodology or a few large claims can have excessive influence on estimated reserves.

Bornhuetter-Ferguson method

1. Calculate the expected ultimate claim payments (using expected ultimate loss ratio times earned premiums)
2. Calculate loss development factors using chain-ladder method
3. Work backwards from expected ultimate payments using loss development factors to get expected loss development.

Question 15

Recall Question 14, where the average loss development factors were

Year	1/0	2/1	3/2	4/3	5/4
Average	1.187962	1.200218	1.11665	1.052196	1.010355

Suppose the expected loss ratio is 0.72, and the earned premiums are

Accident Year	2012	2013	2014	2015	2016
Earned Prem.	180,000	205,000	210,000	220,000	270,000

Use the Bornhuetter-Fergusson method to calculate the loss reserves needed for each accident year.

Loss Reserving for Reinsurance

- Delays in payment generally longer, because reinsurer deals with larger policies and because of the delay in notifying the reinsurer.
- Common clause for excess-of-loss reinsurance: reinsurer notified when incurred loss estimates exceed 50% of attachment point.
- Ceding company must publish direct (without reinsurance) and net reserves. Ceding company legally responsible for claim payments even if it cannot collect its payments from the reinsurer.
- Usually company calculates, direct, net and ceded reserves to compare the approaches and remove inconsistencies.
- Reinsurer will usually group by treaty type and coverage type. May also group by attachment point.
- Catastrophes are known about quickly after the event, but the total reinsured losses will often take a long time to assess. The sudden demand for repairs, etc. can cause inflation in claim costs.

Functions Used to Describe a Distribution

Any Distribution

- Cumulative Distribution Function
- Survival Function
- Moment Generating Function (For light-tailed distributions)

Continuous Distributions

- Probability Density Function
- Hazard Rate Functions (Force of Mortality)

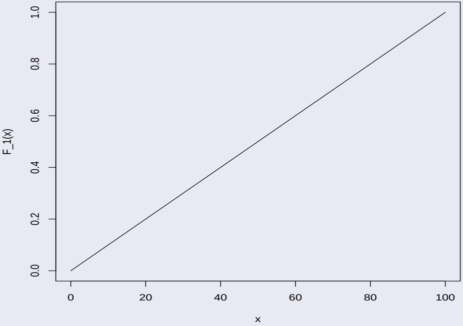
Discrete Distributions

- Probability Mass Function
- Probability Generating Function (for arithmetic distributions)

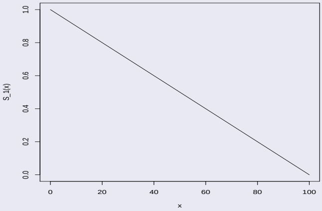
Question 16

Find the survival function, density function and hazard-rate function for the following Model:

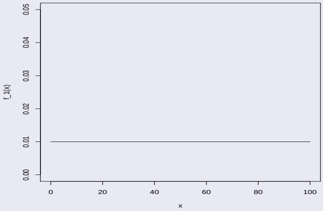
$$F_1(x) = \begin{cases} 0 & x < 0 \\ 0.01x & 0 \leqslant x < 100 \\ 1 & x \geqslant 100 \end{cases}$$



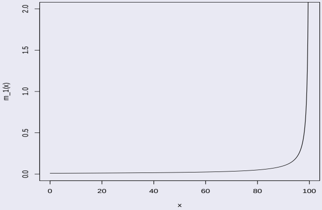
Answer to Question 16



Survival Function



Density Function

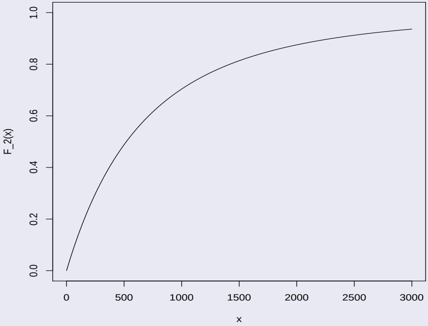


Hazard Rate

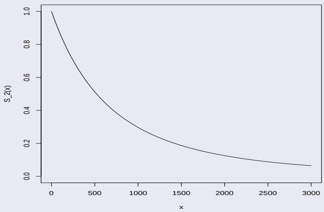
Question 17

Find the survival function, density function and hazard-rate function for the following Model.

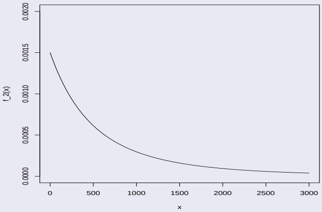
$$F_2(x) = \begin{cases} 0 & x < 0 \\ 1 - \left(\frac{2000}{x+2000}\right)^3 & x \geqslant 0 \end{cases}$$



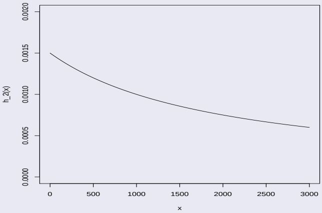
Answer to Question 17



Survival Function



Density Function

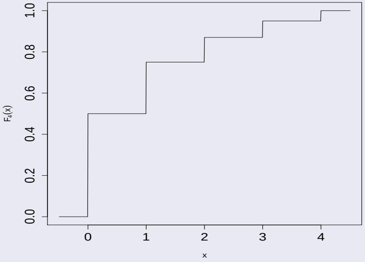


Hazard Rate

Question 18

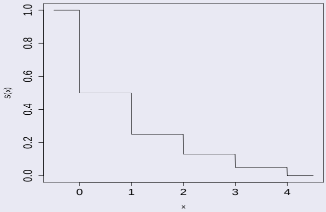
Find the survival function and probability mass function for the following Model.

$$F_3(x) = \begin{cases} 0 & x < 0 \\ 0.5 & 0 \leqslant x < 1 \\ 0.75 & 1 \leqslant x < 2 \\ 0.87 & 2 \leqslant x < 3 \\ 0.95 & 3 \leqslant x < 4 \\ 1 & x \geqslant 4 \end{cases}$$

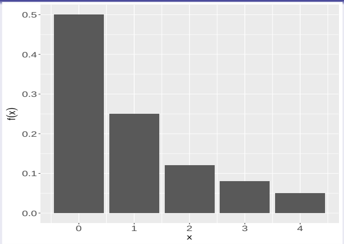


Key Functions and Four Models

Answer to Question 18



Survival Function

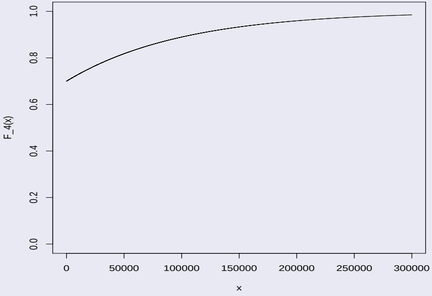


Probability Mass Function

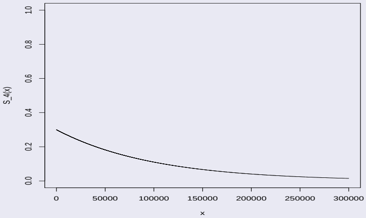
Question 19

Find the survival function, density function and hazard-rate function for the following Model.

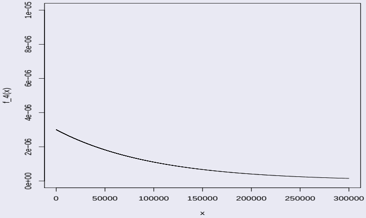
$$F_4(x) = \begin{cases} 0 & x < 0 \\ 1 - 0.3e^{-0.00001x} & x \geqslant 0 \end{cases}$$



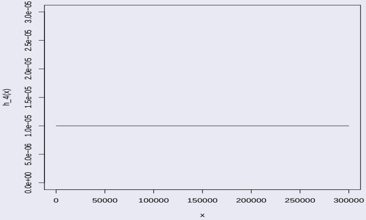
Answer to Question 19



Survival Function



Density Function



3.1 Moments

Raw Moments

The *n*th raw moment of a random variable X is $\mu'_n = \mathbb{E}(X^n)$.

Centralised Moments

The *n*th centralised moment (for $n \geqslant 2$) is $\mu_n = \mathbb{E}\left((X - \mathbb{E}(X))^n\right)$.

Raw Moments

The nth raw moment of a random variable X is $\mu'_n = \mathbb{E}(X^n)$.

Centralised Moments

The nth centralised moment (for $n \geqslant 2$) is $\mu_n = \mathbb{E}\left((X - \mathbb{E}(X))^n\right)$.

Centralised Moments in terms of Raw Moments

$$
\begin{aligned}
\mu_2(x) &= \mathbb{E}((X-\mu_1)^2) = \mathbb{E}(X^2 + (\mu_1)^2 - 2X\mu_1) \\
&= \mathbb{E}(X^2) + (\mu_1)^2 - 2\mu_1\mathbb{E}(X) = \mu'_2 - (\mu_1)^2 \\
\mu_3(x) &= \mathbb{E}((X-\mu_1)^3) = \mathbb{E}(X^3 - 3\mu_1 X^2 + 3(\mu_1)^2 X - (\mu_1)^3) \\
&= \mathbb{E}(X^3) - 3\mu_1\mathbb{E}(X^2) + 3(\mu_1)^2\mathbb{E}(X) - (\mu_1)^3 \\
&= \mu'_3 - 3\mu_1\mu'_2 + 2(\mu_1)^3 \\
\mu_4(x) &= \mathbb{E}((X-\mu_1)^4) = \mathbb{E}(X^4 - 4\mu_1 X^3 + 6(\mu_1)^2 X^2 - 4(\mu_1)^3 X + (\mu_1)^4) \\
&= \mu'_4 - 4\mu_1\mu'_3 + 6(\mu_1)^2\mu'_2 - 3(\mu_1)^4
\end{aligned}
$$

Calculating raw moments from the survival function

If $X \geqslant 0$, $n > 0$.

$$\begin{aligned}\mu_n(X) = \mathbb{E}(X^n) &= \int_0^\infty f(x) x^n \, dx \\ &= [-S(x)x^n]_0^\infty + \int_0^\infty S(x) n x^{n-1} \, dx \\ &= \int_0^\infty S(x) n x^{n-1} \, dx\end{aligned}$$

Coefficient of Variation

The coefficient of variation is $\frac{\sqrt{\mu_2}}{\mu_1}$. This measures the relative variability of a positive random variable.

Skewness

The skewness of a random variable is $\frac{\mu_3}{(\mu_2)^{\frac{3}{2}}}$. For a symmetric distribution, the skewness is 0 if defined.

Kurtosis

The kurtosis of a random variable is $\frac{\mu_4}{(\mu_2)^2}$. This measures the heaviness of the tail. For a normal distribution, the kurtosis is 3.

Question 20

Find the coefficient of variation, skewness and kurtosis of the models in Questions 16–19.

Question 21

A gamma distribution has probability density function given by

$$f(x) = \frac{\left(\frac{x}{\theta}\right)^{\alpha} e^{-\frac{x}{\theta}}}{x\Gamma(\alpha)}$$

where

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx$$

Find the coefficient of variation and skewness of a gamma distribution.

Question 22

Calculate the mean excess loss function $l(x) = \mathbb{E}((X - x)_+)$ for the distribution in Question 17.

$$F_2(x) = \begin{cases} 0 & x < 0 \\ 1 - \left(\frac{2000}{x+2000}\right)^3 & x \geqslant 0 \end{cases}$$

Question 23

Find the median for a gamma distribution with parameters $\alpha = 2$ and $\theta = \frac{e^2}{6}$.

Question 24

Let X be a random variable, and let π_p denote the 100pth percentile of X. Calculate the 100pth percentiles of the excess loss random variable $(X - d)_+$ and limited loss random variable $X \wedge u$ in terms of π_p.

Question 25

A beta distribution has support $[0, 1]$ and probability density function

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx}$$

on its support. Suppose we want to construct a beta distribution such that $\alpha = 2$ and its 95th percentile is 0.8. What should β be?

3.3 Generating Functions and Sums of Random Variables

Moment generating function

$$M_X(t) = \mathbb{E}(e^{tX})$$

- nth derivative $\left(\frac{d}{dt}\right)^n M_X(t)\Big|_{t=0} = \mathbb{E}(X^n)$.
- If X and Y are independent, then $M_{X+Y}(t) = M_X(t)M_Y(t)$.

Probability generating function

$$P_X(z) = \mathbb{E}(z^X)$$

- nth derivative $\left(\frac{d}{dz}\right)^n P_X(z)\Big|_{z=0} = n!P(X = n)$.
- If X and Y are independent, then $P_{X+Y}(z) = P_X(z)P_Y(z)$.

Question 26

An insurance company insures 16 companies. The claims from each company follow a gamma distribution with $\alpha = 1$ and $\theta = 250$. Calculate the probability that the total loss exceeds $6,000.

Question 27

A negative binomial random variable with parameters β and k has probability mass function given by

$$P(X = n) = \binom{r+n-1}{n}\beta^n(1+\beta)^{-(r+n)}$$

[r need not be an integer. The binomial coefficient is given by $\frac{r(r+1)\cdots(r+n-1)}{n!}$. If r is an integer. This is the distribution of the number of failures before r successes, if the probability of success is $p = \frac{1}{1+\beta}$.]
(a) Calculate the probability generating function for a negative binomial with parameters r and β.
(b) What is the distribution of the sum of negative binomial random variables with parameters r_1, β and r_2, β (same value of β)?

3.4 Tail Weight

	Normal (light-tailed)	Pareto (heavy-tailed)
$S(x)$	Quickly $\to 0$	Slowly $\to 0$
$f(x)$	Quickly $\to 0$	Slowly $\to 0$
Outliers	Little influence	Significant influence
CLT	Fast convergence	Slow convergence
Moments	All exist	May not exist
MGF	Exists	Does not exist
Hazard rate	Tends to ∞	Tends to 0

Question 28

A Pareto distribution with parameters α and θ has probability density function

$$f(x) = \frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}}$$

Show that the kth moment exists only for $k < \alpha$.

Question 29

Let X have a Pareto distribution with parameters α and θ. Let Y have a Pareto distributions with parameters $\alpha' > \alpha$ and θ', chosen so that the distributions have the same mean. Show that for any $k > 1$ such that the kth moments of X and Y both exist, the kth moment of X is larger than the kth moment of Y.

Question 30

Let X have a normal distribution with mean μ and variance σ^2. Let Y have a gamma distribution with parameters α and θ. Calculate the limit $\lim_{x\to\infty} \frac{f_Y(x)}{f_X(x)}$. Which distribution has a heavier tail?

Question 31

Let X have a Pareto distribution with parameters α and θ, and let Y have a Pareto distribution with parameters α and θ'. Calculate the limit $\lim_{x\to\infty} \frac{f_Y(x)}{f_X(x)}$. Which distribution has a heavier tail?

Question 32

Calculate the limiting behaviour of the hazard rate functions for a gamma distribution and a Pareto distribution.

Question 33

Calculate the mean excess loss function $l(x) = \mathbb{E}((X - x)_+)$ for a Pareto distribution with parameters α and θ.

3.5 Measures of Risk— Coherence Properties

Subadditivity

$$\rho(X+Y) \leqslant \rho(X)+\rho(Y)$$

Monotonicity

If $P(X \geqslant Y) = 1$, then

$$\rho(X) \geqslant \rho(Y)$$

Positive homogeneity

For any $a > 0$,

$$\rho(aX) = a\rho(X)$$

Translation invariance

For any $c > 0$,

$$\rho(X+c) = \rho(X)+c$$

Standard deviation principle

$$\rho(X) = \mathbb{E}(X) + k\sqrt{\mathrm{Var}(X)}$$

for some constant k.

Value-at-Risk (VaR)

$$\mathrm{VaR}_p(X) = \pi_{X,p}$$

for some level $0 < p < 1$.

Tail Value-at-Risk (TVaR)

Sometimes called Conditional Tail Expectation (CTE).

$$\mathrm{TVaR}_p(X) = \frac{1}{1-p}\int_{\mathrm{VaR}_p(X)}^{\infty} S_X(x)\,dx$$

The expected loss given that the loss exceeds the pth percentile.

Question 34

Which of the coherence properties of risk measures does the standard deviation principle satisfy?

Question 35

The density function of an inverse Pareto distribution is

$$f(x) = \frac{\tau\theta x^{\tau-1}}{(x+\theta)^{\tau+1}}$$

and the distribution function is

$$F(x) = \left(\frac{x}{x+\theta}\right)^{\tau}$$

Find the VaR of this distribution at the level p.

Question 36

Recall that the survival function of a Pareto distribution is

$$S(x) = \left(\frac{\theta}{x+\theta}\right)^{\alpha}$$

The *VaR* of the Pareto distribution at level p is therefore

$$\mathrm{VaR}_p(X) = \theta\left((1-p)^{-\frac{1}{\alpha}} - 1\right)$$

Find the TVaR of this distribution at the level p.

Question 37

Show that TVaR is a coherent risk measure.

Question 38

Calculate the VaR and TVaR of the following distribution:

x	$F(x)$	$F(\lceil x \rceil)$
$0 < x < 1$	$0.16x$	0.16
$1 < x < 2$	$0.34x - 0.18$	0.5
$2 < x < 3$	$0.28x - 0.06$	0.78
$3 < x < 4$	$0.19x + 0.21$	0.97
$4 < x < 5$	$0.026x + 0.866$	0.996
$5 < x < 6$	$0.004x + 0.976$	1

(a) At the 90% level

(b) At the 99% level

Question 39

An insurance company models investment risk as following a gamma distribution with $\theta = 2000$ and $\alpha = 4$. Calculate the TVaR of this distribution at the 95% level.

Question 40

An insurance company models the loss on a particular claim as following a mixture distribution: With probability 0.4, the loss follows a Pareto distribution with $\theta = 1000$, $\alpha = 4$. With probability 0.6, the loss follows a Pareto distribution with $\theta = 1000$ and $\alpha = 8$.
(a) Calculate the risk using the standard deviation principle with $k = 3.5$.
(b) Calculate the VaR and TVaR at the 99% level.

Advantages of Simple Models

- Better estimation.
- Less variance in parameter estimates.

Advantages of Complicated Models

- More accurate description of true distribution.
- Less bias in parameter estimates.

Question 41

(a) Show that the following distributions are scale distributions:

- Exponential
- Gamma
- Normal
- Pareto

(b) Which of the above distributions have scale parameters?

Question 42

Claims follow a Pareto distribution with $\alpha = 2$. Claims experience uniform inflation of 6% per year. Let *r* be the ratio of claims exceeding *d* next year over the proportion of claims exceeding *d* this year. Calculate the limit of *r* as *d* increases to ∞.

Semiparametric and Nonparametric Distributions

Semiparametric Distributions

A variable-component mixture distribution is a mixture distribution where the number of components is not fixed in advance. Often all components of the mixture will have the same distribution (but different parameters).

Data Dependant Distributions

These have as many parameters as data points (or more). As the amount of available data increases, the model becomes more complicated to match.

- The empirical model
- Kernel smoothing models

Question 43

We obtain the following sample of claims (in thousands of dollars): 0.3, 1.2, 1.4, 1.9, 4.7. Using a kernel smoothing model with a Gaussian (normal) kernel with standard deviation 0.8, estimate the probability that the next claim received is more than 3.

Question 44

An insurance company observes the following sample of claims:
2.4, 2.8, 3.5, 3.9, 4.2
It models the distribution of claim sizes using a kernel density model with a uniform kernel with bandwidth 3.
(a) Under this model, what is the median claim size?
(b) What is the median claim size if the company uses a uniform kernel with bandwidth 0.5?

Question 45

An insurance company observes the following sample of claims:
1.2, 1.4, 2.1, 2.9, 4.3
It models the distribution of claim sizes using a kernel density model with a normal kernel with standard deviation 2. Under this model, what is the probability that a claim is greater than 3.5?

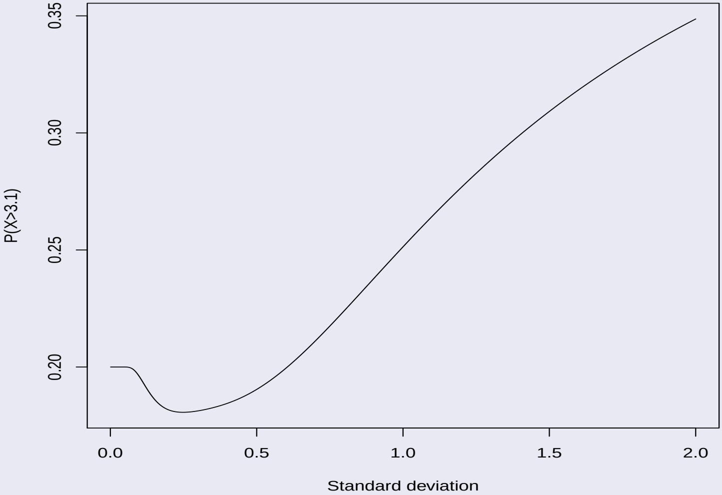
Question 46

An insurance company observes the following sample of claims:
1.8, 2.1, 2.1, 2.4, 3.6
It models the distribution of claim sizes using a kernel density model. The kernel is a gamma distribution with $\alpha = 3$ and mean equal to the observed point, (so $\theta = \frac{x}{3}$). Calculate the probability under this model that a random claim is greater than 5.

Question 47

An insurance company observes the following sample of claims:
1.4, 1.9, 2.0, 2.8, 3.3
It models the distribution of claim sizes using a kernel density model with Gaussian (normal) kernel. Under this model, what is the probability that a random claim is greater than 3.1 for various values of standard deviation of the kernel distribution?

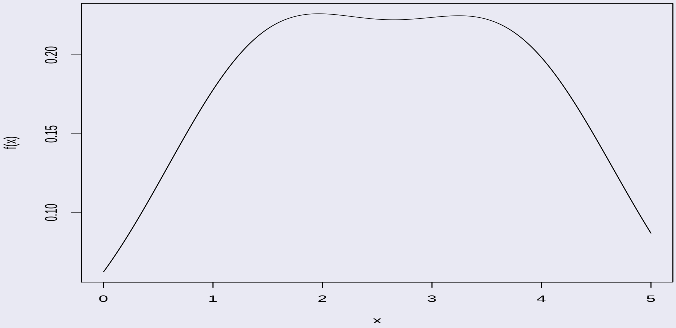
Answer to Question 47



Question 48

An insurance company observes the following sample of claims:
1.4, 1.5, 1.7, 3.5, 3.7, 3.9
It models the distribution of claim sizes using a kernel density model with Gaussian (normal) kernel with standard deviation 1. Under this model, what is the probability density function for the size of a random claim?

Answer to Question 48



$$f(x) = \frac{1}{6\sqrt{2\pi}} \left(e^{-\frac{(x-1.4)^2}{2}} + e^{-\frac{(x-1.5)^2}{2}} + e^{-\frac{(x-1.7)^2}{2}} + e^{-\frac{(x-3.5)^2}{2}} + e^{-\frac{(x-3.7)^2}{2}} + e^{-\frac{(x-3.9)^2}{2}} \right)$$

Methods to Create New Distributions

This Chapter

- Multiplication by a constant
- Raising to a power
- Exponentiation
- Mixing
- Frailty models
- Splicing

More Detail Later

- Truncation
- Censorship
- Modification at Zero

Question 49

Let X have a beta distribution with parameters α and β. Find the probability density function of $5X$.

Question 50

Find the probability density function of the inverse of a Gamma distribution with parameters α and θ.

Question 51

If X follows a normal distribution with mean 0 and variance σ^2, what is the distribution of X^2?

Question 52

A log-normal distribution is the distribution of e^X, where X has a normal distribution with parameters μ and σ^2. Calculate the probability density function of a log-normal distribution.

Question 53

Let X follow a Pareto distribution with parameters α and θ. Let $Y = \log\left(1 + \frac{X}{\theta}\right)$. What is the distribution of Y?

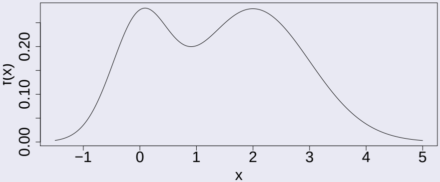
5.2.4 Mixture Distributions

Definition

Let Θ follow a given distribution. A mixture distribution X has a parametric conditional distribution $f_{X|\Theta}$ for each value of Θ.
If Θ can take only a finite number of values, then X is a finite mixture.

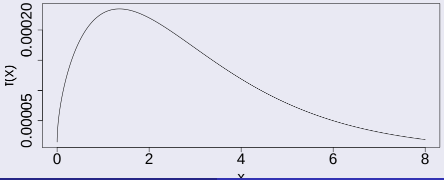
Example

Let Z be a Bernoulli random variable with $P(Z = 1) = p$, and let $X|Z = 1 \sim N(\mu_1, \sigma_1)$ and let $X|Z = 0 \sim N(\mu_2, \sigma_2)$.



Example

Let $A \sim Gamma(\alpha, \theta)$ and let $X|A \sim Gamma(\alpha = A, \theta)$.



Question 54

Suppose that the amount paid on a claim for an insurance policy on a car with value *x* follows an inverse gamma distribution with $\theta = 1000 + 0.1x$ and $\alpha = 5$. Suppose that the value of a randomly selected car follows an exponential distribution with mean $7,000. What is the distribution of the amount paid on a random claim?

Question 55

Given a value of $\Theta = \theta$, the variable X follows an exponential distribution with hazard rate θ. Θ is uniformly distributed on $(1, 11)$. Calculate $S_X(0.5)$.

Question 56

The age at death of a random person is modelled as having a hazard rate given by $0.0001(20 - x)$ for people under the age of 10, then a hazard rate given by $10^{-5}x^2$ for people over the age of 10. Calculate the probability under this model that a randomly chosen person lives to age 75.

Question 57

Each house has a risk factor Θ, which follows a Pareto distribution with $\theta = 1000$ and $\alpha = 2$. Given that a house has risk factor θ, the size of a loss occuring on that home insurance policy follows a Pareto distribution with $\alpha = 3$. What is the probability that a claim on a randomly chosen house exceeds \$1,000.

Question 58

The time until a particular policy results in a claim follows an inverse Weibull distribution with $\tau = 2$ and θ varying between policies. The value of θ for a random policy follows an exponential distribution with mean 4.

(a) What is the probability that a policy produces no claims for 6 years?

(b) Given that a policy has produced no claims for 6 years, what is the probability that it produces no claim the following year?

Question 59

The mortality rate (hazard rate) of an individual aged x is $\lambda e^{0.08x}$, where λ varies between individuals. The value of λ for a random individual at birth follows a gamma distribution with $\theta = 0.00001$ and $\alpha = 3$.
What is the probability that a randomly chosen individual aged 40 survives to age 90?

Question 60

An insurance company divides claims under a certain policy into "small claims" and "large claims", where it defines "small claims" as claims with value under \$2,000, and "large claims" as claims over \$2,000. It models small claims as following a truncated gamma distribution with $\alpha = 3$ and $\theta = 1000$, and models large claims as following a (truncated) Pareto distribution with $\alpha = 4$ and $\theta = 3000$. It wants the overall distribution to have a continuous density function. What proportion of small and large claims must it set in order to ensure this?

Question 61

Prove the Law of Total Variance: For any random variables X and Z

$$\mathrm{Var}(X) = \mathbb{E}(\mathrm{Var}(X|Z)) + \mathrm{Var}(\mathbb{E}(X|Z))$$

Question 62

An insurance company models the cost of a claim for a particular policyholder as following a Pareto distribution with $\alpha = 3$ and θ depending on the policyholder. For a random policyholder, this θ follows a gamma distribution with $\alpha = 2$ and $\theta = 3000$. What is the variance of the cost of a random claim?

Question 63

An insurance company models the cost of a claim as following an inverse gamma distribution with $\alpha = 2$ and θ varying between policies. For a random policy, θ follows a gamma distribution with $\theta = 1000$ and $\alpha = 2$. Calculate the VaR and TVaR of a random policy at the 95% level.

Question 64

Let X be a mixture of 3 distributions:

- with probability p_1, it has mean μ_1 and variance ${\sigma_1}^2$.
- with probability p_2, it has mean μ_2 and variance ${\sigma_2}^2$.
- with probability p_3, it has mean μ_3 and variance ${\sigma_3}^2$.

What are the mean and variance of X?

Question 65

A car insurance company deals with three types of claim:

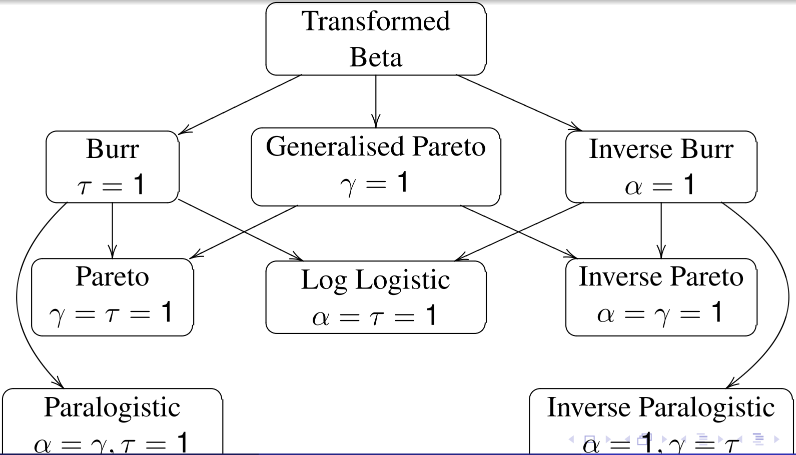
- 15% are for thefts. The claim amount in this case follows an inverse Pareto distribution with $\tau = 3$ and $\theta = 4000$.
- 75% are for collisions. The claim amount in this case follows a Gamma distributions with $\alpha = 50$ and $\theta = 100$.
- 10% are other claims. The claim amount in this case follows an exponential distribution with mean \$3,000.

What proportion of claims are over \$10,000?

The Transformed Beta Family

Transformed Beta Distribution

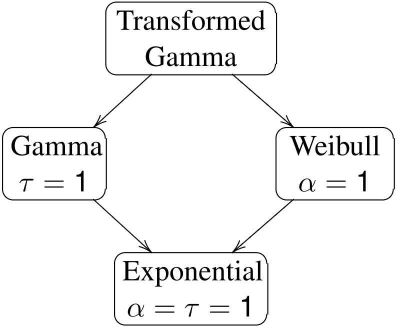
$$f_X(x) = \left(\frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)} \right) \frac{\gamma \left(\frac{x}{\theta}\right)^{\gamma\tau}}{x \left(1 + \left(\frac{x}{\theta}\right)^{\gamma}\right)^{\alpha+\tau}}$$



The Transformed Gamma and Inverse Transformed Gamma Families

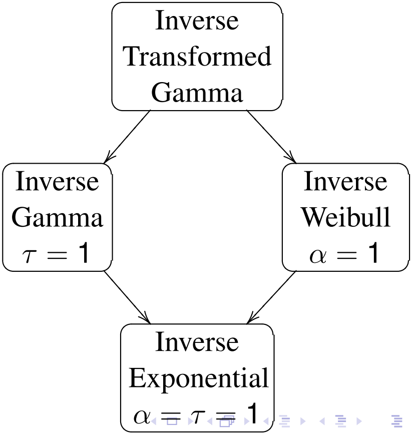
Transformed Gamma

$$f_X(x) = \frac{\tau\left(\frac{x}{\theta}\right)^{\alpha\tau} e^{-\left(\frac{x}{\theta}\right)^{\tau}}}{x\Gamma(\alpha)}$$



Inverse Transformed Gamma

$$f_X(x) = \frac{\tau\left(\frac{\theta}{x}\right)^{\alpha\tau} e^{-\left(\frac{\theta}{x}\right)^{\tau}}}{x\Gamma(\alpha)}$$



Transformed Gamma Family

- Transformed Gamma (generalised gamma)

$$f_X(x) = \frac{\tau \left(\frac{x}{\theta}\right)^{\alpha\tau} e^{-\left(\frac{x}{\theta}\right)^{\tau}}}{x\Gamma(\alpha)}$$

- Gamma ($\tau = 1$)

$$f_X(x) = \frac{\left(\frac{x}{\theta}\right)^{\alpha} e^{-\left(\frac{x}{\theta}\right)}}{x\Gamma(\alpha)}$$

- Weibull ($\alpha = 1$)

$$f_X(x) = \frac{\tau \left(\frac{x}{\theta}\right)^{\tau} e^{-\left(\frac{x}{\theta}\right)^{\tau}}}{x}$$

- Exponential ($\tau = 1$, $\alpha = 1$)

$$f_X(x) = \frac{e^{-\frac{x}{\theta}}}{\theta}$$

Question 66

Let X have a transformed beta distribution with parameters α, γ, θ and τ. Show that $\frac{1}{X}$ also follows a transformed beta distribution, and calculate its parameters.

Question 67

Show that the inverse transformed gamma distribution is the limit of a transformed beta distribution as $\tau \to \infty$, $\theta \to 0$ and $\theta\tau^{\frac{1}{\gamma}} \to \xi$.

Question 68

Show that the limiting distibution of the transformed gamma distribution as $\alpha \to \infty$, $\tau \to 0$ and $\theta \to 0$, with $\frac{\sqrt{\theta^\tau}}{\tau} \to \sigma$ and $\frac{\theta^\tau \alpha - 1}{\tau} \to \mu$ is a lognormal distribution.

Question 69

Show that a distribution from the linear exponential family with pdf

$$f_X(x) = \frac{p(x)e^{r(\theta)x}}{q(\theta)}$$

has mean $\mu(\theta) = \frac{q'(\theta)}{r'(\theta)q(\theta)}$ and variance $\frac{\mu'(\theta)}{r'(\theta)}$.

Question 70

Find the mean and variance of a gamma distribution.

Question 71

(a) Calculate the probability generating function of a Poisson random variable.
(b) Show that a sum of independent Poisson random variables is a Poisson random variable.

Question 72

Let the number of losses experienced for a particular insurance product follow a Poisson distribution with parameter λ. Let each loss lead to a claim with probability p. Show that the number of claims follows a Poisson distribution with parameter λp, and that the number of losses which do not lead to claims is independent of the number of claims.

Question 73

An insurance company takes out 10 different reinsurance policies. Each one has probability 0.2 of leading to a claim.
(a) What is the probability that the company does not need to make any claims to the reinsurers?
(b) What is the probability that the number of reinsurance claims made is 3?

Question 74

Find the probability generating function of a binomial random variable.

Negative Binomial Distribution

$$P(X = k) = \binom{k+r-1}{k} \left(\frac{\beta}{1+\beta} \right)^k \left(\frac{1}{1+\beta} \right)^r$$

Question 75

Show that a gamma mixture of Poisson random variables gives a negative binomial random variable, and calculate the parameters

Question 76

An insurance company insures a large number of houses. The total number of claims resulting from these policies in a given month is modelled as a negative binomial distribution, and the company estimates that $r = 70$ and $\beta = 0.08$. What is the probability that at least one of these policies leads to a claim in the first month?

Question 77

Find the probability generating function of a negative binomial random variable.

Question 78

Show that the limiting case of the negative binomial distribution as $r \to \infty$ and $\beta \to 0$ with $r\beta \to \lambda$, is a Poisson distribution with parameter λ.

$(a, b, 0)$-class

A distribution is in the $(a, b, 0)$ class if its probability function satisfies

$$\frac{P(X = k)}{P(X = k - 1)} = a + \frac{b}{k}$$

for all $k \geqslant 1$.

Question 79

Show that the binomial, Poisson and Negative binomial distributions are in the $(a, b, 0)$ class, and calculate the values of a and b in each case.

Show that these are the only distributions from the $(a, b, 0)$ class.

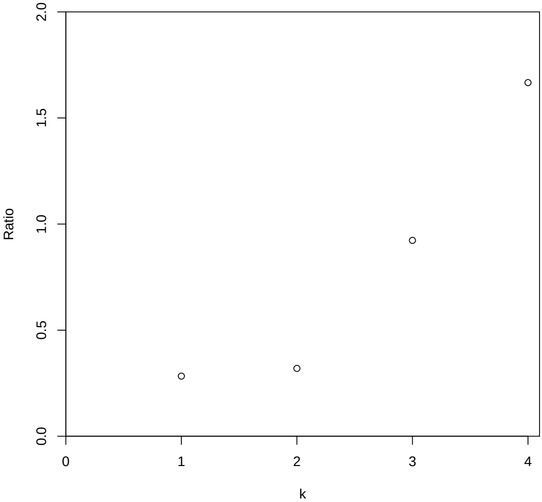
Question 80

The number of claims on each insurance policy over a given time period is observed as follows:

Number of claims	Number of policies
0	861
1	122
2	13
3	3
4	1
5 or more	0

Which distributions from the $(a, b, 0)$-class appear most appropriate for modelling this data?

Plot of $k\frac{n_k}{n_{k-1}}$ against k



6.6 Truncation and Modification at zero

Truncation

Sometimes small (e.g. zero) values for a random variable are not observed. For example, if a loss is less than the deductible, then it is not reported to the insurer, so does not appear in the dataset. This is known as trunctation.

Modification at zero

Sometimes frequency distributions are not truncated, but the probability of zero may be adjusted. For example, some insureds may have other coverage, so may never make claims. This means the probability of zero claims may not follow the same patterns as other frequencies.

$(a, b, 1)$-class

A distribution is in the $(a, b, 1)$ class if its probability function satisfies

$$\frac{P(X = k)}{P(X = k - 1)} = a + \frac{b}{k}$$

for all $k \geqslant 2$.

Relation to $(a, b, 0)$ class

If a and b are valid parameters for a distribution from the $(a, b, 0)$ class, then the distribution from the $(a, b, 1)$ class is a modified $(a, b, 0)$ distribution.
If $a > 0$ and $-2a < b \leqslant -a$, then we do not get a distribution in the $(a, b, 0)$ class, because $p_1 = (a + b)p_0$ can't work. However, we can get a distribution from the $(a, b, 1)$ class.

Question 81

The number of claims resulting from a given insurance policy follows a negative binomial distribution with $r = 3.4$ and $\beta = 0.06$. A company is looking only at the policies that result in at least one claim. If it selects one of these policies, what is the probability that it has resulted in at least 3 claims?

Question 82

The number of claims an individual would make under a certain dental insurance policy in a year follows a negative binomial distribution with $r = 0.8$ and $\beta = 0.7$. However, individuals who would make no claims are less likely to buy the policy. An individual who would make no claims has only a 0.5 probability of buying the policy (but all other individuals buy the policy). What is the distribution of the number of claims made by an individual policyholder.

Question 83

(a) Show that for an extended truncated negative binomial distribution with parameters $r \neq 0$ and β,

$$p_1 = \frac{r\beta}{(1+\beta)((1+\beta)^r - 1)}$$

(b) Calculate the expectation of an extended truncated negative binomial distribution with $r \neq 0$.

(c) Show that for an extended truncated negative binomial distribution with parameters $r = 0$ and β,

$$p_1 = \frac{\beta}{(1+\beta)\log(1+\beta)}$$

(This distribution is called a logarithmic distribution.)

(d) Calculate the expectation of a logarithmic distribution.

Question 84

(a) Calculate the probabilites and expected value of an extended truncated negative binomial distribution with $r = -0.6$ and $\beta = 0.8$.
(b) Calculate the probabilites and expected value of a logarithmic distribution (ETNB with r=0) with $\beta = 0.5$.

Deductibles (Revision)

If a policy has a deductible d, then the amount paid for a loss X is $(X - d)_+$.

Dealing with Deductibles

- Deductibles reduce claim frequency.
- For severity distribution, we sometimes consider per loss, and sometimes per claim.
- Deductibles always reduce per loss severity, but might increase or decrease per claim severity.

Question 85

The size of loss under a car insurance policy follows a Burr distribution with $\gamma = 0.8$, $\alpha = 1.4$ and $\theta = 3000$. The insurance company is considering adding a deductible of 1000 to the policy.
(a) What is the new distribution of the payment per claim?
(b) If the company introduces a franchise deductible instead, what is the expected value of a claim made under the policy?

Loss Elimination Ratio

The Loss Elimination Ratio is the ratio

$$\frac{\text{Losses paid by policyholder due to deductible}}{\text{Total losses}} = 1 - \frac{\text{Losses paid by insurer}}{\text{Total losses}}$$

Question 86

The severity of a loss on a health insurance policy follows an inverse gamma distribution with $\alpha = 3.2$ and $\theta = 2000$. Calculate the loss elimination ratio of a deductible of $500.

Question 87

Losses have a Pareto distribution with $\alpha = 2$ and $\theta = k$. There is a deductible of $2k$. Determine the loss elimination ratio before and after inflation of 100%.

8.4 Policy Limits

Policy Limits (Revision)

If a policy has a limit u, then the amount paid for a loss X is $X \wedge u$.

Dealing with Limits

- Deductibles are like limits from the point of view of the policyholder — with a deductible d, the policyholder pays $X \wedge d$ and the insurer pays $(X - d)_+$ With a limit u, the insurer pays $X \wedge u$ and the policyholder pays $(X - u)_+$.
- Limits decrease the effect of inflation.

Question 88

Losses on a particular insurance policy follow a Weibull distribution with $\theta = 3000$ and $\tau = 3$.
(a) What is the expected loss?
(b) What is the expected loss if a policy limit of \$5,000 is imposed on claims.
(c) What is the percentage increase in the expected loss on this policy if there is 20% inflation?

Question 89

Losses follow a Pareto distribution with $\alpha = 2$ and $\theta = 5,000$. An insurance policy pays the following for each loss. There is no insurance payment for the first 1,000. For losses between 1,000 and 6,000, the insurance pays 80%. Losses above 6,000 are paid by the insured until the insured has paid a total payment of 10,000. For any remaining part of the loss, the insurance pays 90%. Determine the expected insurance payment per loss.

Question 90

The loss severity random variable X has an exponential distribution with mean 10,000. Determine the coefficient of variation of the variables per payment claim amount Y^P and per loss claim amount Y^L, based on $d = 30,000$.

Advantages of Modelling Number of Losses and Severities Separately

- Dealing with changes to exposure (e.g. number of policies)
- Dealing with inflation
- Dealing with changes to individual policies
- Understanding the impact of changing deductibles on claim frequencies.
- Combining data with a range of different deductibles and limits can give a better picture of the loss distribution.
- Consistency between models of non-covered losses to insureds, claims to insurers, and claims to reinsurers.
- The effect of the shapes of separate distributions of number and severity give an indicator of how each influences the overall aggregate loss.

Model Choices

Practical Considerations

- Scale parameters for severity allow for change of currency or inflation.
- For frequency, models with pgf $P(z;\alpha) = Q(z)^\alpha$ can deal with changes to number of policies sold, or time period.
- Modification at zero prevents infinite divisibility. However, modification at zero may still be appropriate.

Question 91

Which discrete distributions satisfy

$$P(z; \alpha) = Q(z)^{\alpha}$$

for some parameter α?

Question 92

Calculate the first three moments of a compound model.

Question 93

When an individual is admitted to hospital, the distribution of charges incurred are as described in the following table:

charge	mean	standard deviation
Room	1000	500
other	500	300

The covariance between room charges and other charges is 100,000. An insurer issues a policy which reimburses 100% for room charges and 80% for other charges. The number of hospital admissions has a Poisson distribution with parameter 4. Determine the mean and standard deviation for the insurer's payout on the policy.

Question 94

Aggregate payments have a compound distribution. The frequency distribution is negative binomial with $r = 16$, $\beta = 6$. The severity distribution is uniform on the interval $(0, 8)$. Using a normal approximation, determine the premium such that there is a 5% probability that aggregate payments exceed the premium.

Question 95

For a group health contract, aggregate claims are assumed to have an exponential distribution with mean θ estimated by the group underwriter. Aggregate stop-loss insurance for total claims in excess of 125% of the expected claims, is provided for a premium of twice the expected stop-loss claims. It is discovered that the expected total claims value used was 10% too low. What is the loading percentage on the stop-loss policy under the true distribution?

Question 96

In a group life insurance policy, a life insurance company insures 10,000 individuals at a given company. It classifies these workers in the following classes:

Type of worker	number	average annual probability of dying	death benefit
Manual Laborer	4,622	0.01	$100,000
Administrator	3,540	0.002	$90,000
Manager	802	0.01	$200,000
Senior Manager	36	0.02	$1,000,000

What is the probability that the aggregate benefit paid out in a year exceeds $10,000,000?

Question 97

Using the same data as in Question 96, estimate the probability by modelling the distribution of the aggregate risk as:
(a) a normal distribution
(b) a gamma distribution
(c) a log-normal distribution

Question 98

An insurance company has the following portfolio of car insurance policies:

Type of driver	Number	Probability claim	mean of claim	standard deviation
Safe drivers	800	0.02	\$3,000	\$1,500
Average drivers	2100	0.05	\$4,000	\$1,600
Dangerous drivers	500	0.12	\$5,000	\$1,500

(a) Using a gamma approximation for the aggregate losses on this portfolio, calculate the cost of reinsuring losses above \$800,000, if the loading on the reinsurance premium is one standard deviation above the expected claim payment on the reinsurance policy.

(b) How much does the premium change if we use a normal approximation?

Question 99

An insurance company assumes that for smokers, the claim probability is 0.02, while for non-smokers, it is 0.01. A group of mutually independant lives has coverage of 1000 per life. The company assumes that 20% of lives are smokers. Based on this assumption, the premium is set equal to 110% of expected claims. If 30% of the lives are smokers, the probability that claims will exceed the premium is less than 0.2. Using a normal approximation, determine the minimum number of lives in the group.

Example

- An insurance company offers group life insurance to all 372 employees of a company.
- The premium is set at \$1,000 per year.
- The company notices that the average annual total claim over the past 7 years is \$126,000 — Far lower than the total premiums charged.

The company contacts the insurers and asks for a reduction in premiums on the basis that premiums are much larger than the average claim.

(a) Is this request reasonable?

(b) What would be a fair reduction in premium?

Definition

We assign full credibility to a policyholder's past history if we have sufficient data to use the policyholder's average claim for our premium estimate.

Criterion for Full Credibility

Let ξ be the (unknown) expected claim from a policyholder. We pick $r \geqslant 0$ and $0 < p < 1$. We assign full credibility to X if

$$P(|\overline{X} - \xi| < r\xi) > p$$

That is if with probability p, the relative error of $\overline{X}$ as an estimator for ξ is less than r.

Question 100

Recall our earlier example:

- An insurance company offers group life insurance to all 372 employees of a company.
- The premium is set at $1,000 per year.
- The average annual total claim over the past 7 years is $126,000.

Suppose that all policies have a death benefit of $98,000, and deaths of each employee are independent.
(a) Should the insurers assign full credibility to this experience? (Use $r = 0.05$ and $p = 0.95$.)
(b) How many years of past history are necessary to assign full credibility?

Question 101

Recall our earlier example:

- An insurance company offers group life insurance to all 372 employees of a company.
- The premium is set at $1,000 per year.
- The average annual total claim over the past 7 years is $1,260,000.

Suppose that all policies have a death benefit of $98,000, and deaths of each employee are independent.
If the standard for full credibility is measured in terms of number of claims, instead of number of years, what standard is needed in this case, and how does the standard vary with number of years.

Question 102

A car insurance company is reviewing claims from a particular brand of car. It finds that over the past 3 years:

- it has issued 41,876 annual policies for this type of car.
- The average annual aggregate claim per policy is $962.14.
- The standard deviation of annual aggregate claim per policy is $3,605.52

(a) Should it assign full credibility to the historical data from this type of car?
(b) How many policies would it need in order to assign full credibility?

Question 103

Recall our original example:

- Group life insurance for 372 employees of a company.
- The premium is set at $1,000 per year.
- The average annual total claim over the past 7 years is $126,000.

All policies have a death benefit of $98,000, and deaths of each employee are independent.
In Question 100, we determined that this was not sufficient to assign full credibility to the data, and that 1191.034 years of claims data would be needed for full credibility.

How much credibility should we assign to this data, and what should the resulting premium be?

Question 104

For a particular insurance policy, the average claim is \$230, and the average claim frequency is 1.2 claims per year. A policyholder has enrolled in the policy for 10 years, and has made a total of 19 claims for a total of \$5,822. Calculate the new premium for this policyholder if the standards for full credibility are:

(a) 421 claims for claim frequency, 1,240 claims for severity.

(b) 1146 claims for claim frequency, 611 claims for severity.

(c) 400 years for aggregate losses

17.5 Problems with this Approach

Problems

- No theoretical justification.
- Need to choose r and p arbitrarily.
- Doesn't take into account uncertainty in the book premium.

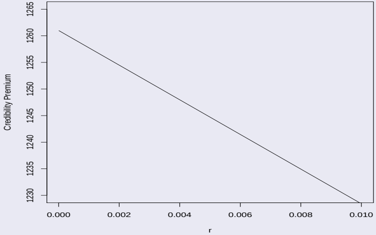
Question 105

An insurance company sells car insurance. The standard annual premium is $1,261. A car manufacturer claims that a certain model of its cars is safer than other cars and should receive a lower premium. The insurance company has issued 3,722 policies for this model of car. The total aggregate claims on these polices were $3,506,608. The variance of the annual aggregate claims on a policy is 8,240,268. Calculate the Credibility premium for different values of r and p.

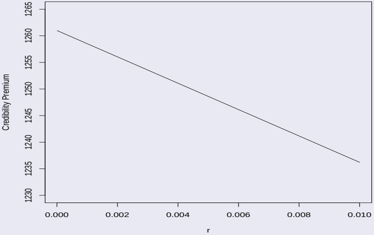
R-code for Question 105

```
#Limited Fluctuation Credibility Premium as r changes
p<-0.05
r<-(1:1000)/100000
Z<- 20.02297*r/qnorm(1-p/2)
Z<-pmin(Z,1)
plot(r,Z*3506608/3722+(1-Z)*1261,type='l')
pdf("LFCredibilityChangeRPp=0.05.pdf")
plot(r,Z*3506608/3722+(1-Z)*1261,type='l',ylab="
    Credibility␣Premium")
dev.off()
```

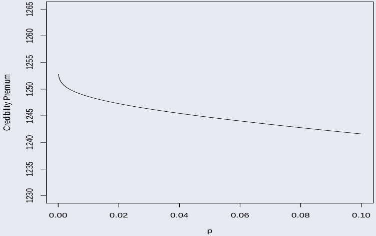
Answer to Question 105



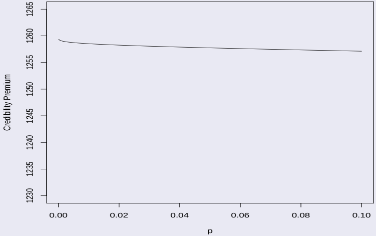
$p = 0.05$



$p = 0.01$



$r = 0.005$



$r = 0.001$

Answer to Question 105