ACSC/STAT 3703, Actuarial Models I

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Homework Sheet 3

Model Solutions

Basic Questions

1. *A distribution has hazard rate $\lambda(x) = x + \frac{3}{6+x}$ for $x \geqslant 0$. Calculate its density function.*

   The survival function is

$$\begin{aligned}S(x) &= e^{-\int_0^x \lambda(s)\,ds} = e^{-\int_0^x x+\frac{3}{6+y}\,dy}\\ &= e^{-\left[\frac{y^2}{2}+3\log(6+y)\right]_0^x}\\ &= e^{-\left(\frac{x^2}{2}+3(\log(6+x)-\log(6))\right)}\\ &= \frac{6^3}{(6+x)^3}e^{-\frac{x^2}{2}}\end{aligned}$$

   The density function is negative the derivative of the survival function. That is

$$\begin{aligned}f(x) &= -\frac{dS(x)}{dx}\\ &= -\frac{d}{dx}\left(\frac{6^3}{(6+x)^3}e^{-\frac{x^2}{2}}\right)\\ &= \frac{3\times 6^3}{(6+x)^4}e^{-\frac{x^2}{2}} + \frac{6^3x}{(6+x)^3}e^{-\frac{x^2}{2}}\\ &= \frac{6^3(3+6x+x^2)}{(6+x)^4}e^{-\frac{x^2}{2}}\end{aligned}$$

2. *A continuous random variable has moment generating function given by $M(t) = \frac{1}{(1-t)^2} + \frac{1}{(1-t^2)^4}$. What is the skewness of the distribution?*

We calculate

$$
\begin{aligned}
M'(t) &= \frac{2}{(1-t)^3} + \frac{8t}{(1-t^2)^5} \\
M''(t) &= \frac{6}{(1-t)^4} + \frac{8}{(1-t^2)^5} + \frac{80t^2}{(1-t^2)^6} \\
M'''(t) &= \frac{24}{(1-t)^5} + \frac{240t}{(1-t^2)^5} + \frac{960t^3}{(1-t^2)^6} \\
M'(0) &= 2 \\
M''(0) &= 14 \\
M'''(0) &= 24
\end{aligned}
$$

Thus $\mathbb{E}(X) = 2$ and $\mathbb{E}(X^2) = 14$ and $\mathbb{E}(X^3) = 24$. This gives $\mathrm{Var}(X) = 14 - 2^2 = 10$ and

$$\mu_3 = \mathbb{E}(X^3) - 3\mathbb{E}(X)\mathbb{E}(X^2) + 2\mathbb{E}(X)^3 = 24 - 3 \times 2 \times 14 + 2 \times 2^3 = -44$$

Thus, the skewness is $\frac{-44}{10^{1.5}} = -1.39140217047$

3. *Calculate the mean excess loss function for a distribution with hazard rate given by* $\lambda(x) = \frac{x}{x+1}$ *for* $x \geqslant 0$.

   Rewriting $\lambda(x) = 1 - \frac{1}{x+1}$, we have

   $$S(x) = e^{-\int_0^x \lambda(y)\,dy} = e^{-\int_0^x 1-\frac{1}{y+1}\,dy} = e^{-x+\log(x+1)} = (x+1)e^{-x}$$

   The mean excess loss function is given by

   $$
   \begin{aligned}
   \mathbb{E}((X-d)_+) &= \int_d^\infty S(x)\,dx \\
   &= \int_d^\infty (x+1)e^{-x}\,dx \\
   &= \left[-(x+1)e^{-x}\right]_d^\infty + \int_d^\infty e^{-x}\,dx \\
   &= (d+1)e^{-d} + e^{-d} \\
   &= (d+2)e^{-d}
   \end{aligned}
   $$

4. *Calculate the probability generating function of a discrete distribution with p.m.f. given by*

   $$f(n) = \frac{e^{-1}}{2}\frac{n^2}{n!}$$

The probability generating function is given by

$$
\begin{aligned}
P(z) = \mathbb{E}(z^X) &= \sum_{n=0}^{\infty} f(x) z^x \\
&= \frac{e^{-1}}{2} \sum_{n=0}^{\infty} \frac{n^2 z^n}{n!} \\
&= \frac{e^{-1}}{2} \sum_{n=1}^{\infty} \frac{n z^n}{(n-1)!} \\
&= \frac{e^{-1}}{2} \sum_{n=1}^{\infty} \frac{(n-1+1) z^n}{(n-1)!} \\
&= \frac{e^{-1}}{2} \left(\sum_{n=1}^{\infty} \frac{z^n}{(n-1)!} + \sum_{n=2}^{\infty} \frac{z^n}{(n-2)!} \right) \\
&= \frac{e^{-1}}{2} \left(z \sum_{m=0}^{\infty} \frac{z^m}{m!} + z^2 \sum_{m=0}^{\infty} \frac{z^m}{m!} \right) \\
&= \frac{z + z^2}{2} e^{z-1}
\end{aligned}
$$

Standard Questions

5. *The total cost of handling a claim is $X_1 + X_2 + Y$ where X_1 and X_2 are i.i.d. discrete non-negative random variables with probability generating function $P_X(z) = \frac{z+1}{z+3}$ and Y is a discrete non-negative random variable with probability generating function $P_Y(z) = \frac{3^z}{4(z+1)}$, independent of X_1 and X_2. What is the moment generating function of $X_1 + X_2 + Y$?*

We have

$$
\begin{aligned}
P_{X_1+X_2+Y}(z) &= P_{X_1}(z) P_{X_2}(z) P_Y(z) \\
&= P_X(z)^2 P_Y(z) \\
&= \left(\frac{z+1}{z+3} \right)^2 \frac{3^z}{4(z+1)} \\
&= \frac{(z+1) 3^z}{4(z+3)^2}
\end{aligned}
$$

The moment generating function is given by $M_X(t) = P_X(e^t)$, so

$$
M_{X_1+X_2+Y}(t) = P_{X_1+X_2+Y}(e^t) = \frac{(e^t+1) 3^{e^t}}{4(e^t+3)^2}
$$

6. *An insurance company is considering two models for its data. The first is a Pareto distribution with survival function*

$$S(x) = \left(\frac{\theta}{\theta + x}\right)^{\alpha}$$

The second is a Weibull distribution with survival function

$$S(x) = e^{-\left(\frac{x}{\theta}\right)^{\tau}}$$

They find that for the fitted parameters, both distributions have the same values of θ, and the same values for the 90th and 95th percentiles. Which distribution has a higher 99th percentile?

[You should get an equation for one of the unknown parameters α or τ. You can numerically solve this equation by trying a range of values and seeing which satisfies the equation.]

The pth percentiles of the Pareto and Weibull distributions are obtained by solving

$$\begin{aligned}
\left(\frac{\theta}{\theta + x}\right)^{\alpha} &= 1 - p \\
x &= \theta(1-p)^{-\alpha} - 1 \\
e^{-\left(\frac{x}{\theta}\right)^{\tau}} &= 1 - p \\
x &= \theta(-\log(1-p))^{\frac{1}{\tau}}
\end{aligned}$$

Setting the 90th and 95th percentiles equal gives:

$$\begin{aligned}
\theta(-\log(0.1))^{\frac{1}{\tau}} &= \theta(0.1^{-\alpha} - 1) \\
\theta(-\log(0.05))^{\frac{1}{\tau}} &= \theta(0.05^{-\alpha} - 1) \\
\frac{\log(-\log(0.1))}{\tau} &= \log\left((0.1)^{-\alpha} - 1\right) \\
\frac{\log(-\log(0.05))}{\tau} &= \log\left((0.05)^{-\alpha} - 1\right) \\
\frac{\log\left((0.1)^{-\alpha} - 1\right)}{\log\left((0.05)^{-\alpha} - 1\right)} &= \frac{\log(-\log(0.1))}{\log(-\log(0.05))} \\
&= 0.760154060074 \\
\log\left((0.1)^{-\alpha} - 1\right) &= 0.760154060074 \log\left((0.05)^{-\alpha} - 1\right) \\
(0.1)^{-\alpha} - 1 &= \left((0.05)^{-\alpha} - 1\right)^{0.760154060074}
\end{aligned}$$

Numerically, we see that this is solved by $\alpha = 1.3228$. Substituting this for α, we get

$$\begin{aligned}\frac{\log(-\log(0.1))}{\tau} &= \log\left((0.1)^{-1.3228}-1\right)\\ \frac{0.834032445246}{\tau} &= 2.9971362061\\ \tau &= \frac{0.834032445246}{2.9971362061}\\ \tau &= 0.2783\end{aligned}$$

The 99th percentiles are

$$\theta\left((0.01)^{-1.3228}-1\right) = 441.1809210074\theta$$

for the Pareto distribution, and

$$\theta(-\log(0.01))^{\frac{1}{0.278276457222}} = 241.7718113347\theta$$

for the Weibull distribution, so the percentile is larger for the Pareto distribution.

Bonus Questions

7. *X and Y are continuous random variables with moment generating functions $M_X(t) = \frac{864}{(t-4)(t-6)^3}$ and $M_Y(t) = \frac{e^{-t^2}}{t+1}$. You are given that X and $X+Y$ are independent. What is the probability generating function of $X+Y$?*

Since X and $X+Y$ are independent, $-X$ and $X+Y$ are independent. so $M_Y(t) = M_{-X}(t)M_{X+Y}(t)$. We have

$$M_{-X}(t) = \mathbb{E}\left(e^{t(-X)}\right) = \mathbb{E}\left(e^{-tX}\right) = M_X(-t) = \frac{864}{(t+4)(t+6)^3}$$

so we have

$$\begin{aligned}\frac{e^{-t^2}}{t+1} &= \frac{864}{(t+4)(t+6)^3}M_{X+Y}(t)\\ M_{X+Y}(t) &= -\frac{(t+4)(t+6)^3e^{-t^2}}{864(t+1)}\end{aligned}$$

The probability generating function satisfies $P(z) = M(\log(z))$, so

$$P_{X+Y}(z) = -\frac{(\log(z)+4)(\log(z)+6)^3e^{-\log(z)^2}}{864(\log(z)+1)}$$