ACSC/STAT 3703, Actuarial Models I

WINTER 2025
Toby Kenney

Homework Sheet 4

Model Solutions

Basic Questions

1. *A distribution has survival function*

$$S(x) = e^{-e^x}$$

for $x \geqslant 0$. How does the tail weight of this distribution compare to that of a normal distribution with $\mu = 0$ and $\sigma^2 = 1$, when tail-weight is assessed by

(a) Asymptotic behaviour of hazard rate.

We differentiate $S(x)$ to get

$$f(x) = e^{x-e^x}$$

so

$$\lambda(x) = \frac{f(x)}{S(x)} = e^x$$

For the normal distribution, we have $S(x) = \Phi(-x)$ and $f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, so $\lambda(x) = \frac{e^{-\frac{\log(x)^2}{2}}}{\sqrt{2\pi}\Phi(-x)}$. Taking the ratio of hazard rates gives

$$\frac{e^{-x-\frac{x^2}{2}}}{\sqrt{2\pi}\Phi(-x)}$$

Letting $u = \frac{1}{x}$, the ratio of hazard rates becomes

$$\frac{e^{-u^{-1}-\frac{u^{-2}}{2}}}{\sqrt{2\pi}\Phi(-u^{-1})}$$

We want to take the limit as $u \to 0$. By l'Hôpital's rule, this limit is

$$\lim_{u\to 0} \frac{\frac{d}{du}e^{-u^{-1}-\frac{u^{-2}}{2}}}{\frac{d}{du}\sqrt{2\pi}\Phi(-u^{-1})} = \lim_{u\to 0} \frac{e^{-u^{-1}-\frac{u^{-2}}{2}}\left(u^{-2}-u^{-3}\right)}{u^{-2}e^{-\frac{u^{-2}}{2}}} = \lim_{u\to 0} e^{-u^{-1}}\left(1-u^{-1}\right) = 0$$

so the normal distribution has a heavier tail.

(b) Existence of moments.

For the normal distribution, the moment generating function exists for all t, and therefore all finite moments exist. For the given distribution, the moment generating function is

$$M_X(t) = \mathbb{E}(e^{tX}) = \int_0^\infty e^{tx}e^x e^{-e^x}\,dx = \int_0^\infty e^{(t+1)x-e^x}\,dx$$

It is easy to see that $e^{(t+1)x-e^x} \to 0$ very quickly for large x, so the integral converges for all t, meaning that both distributions have moment generating functions defined for all $t \in \mathbb{R}$, so we cannot use existence of moments to determine which distribution has the heavier tail.

2. *Which coherence properties are satisfied by the following measure of risk?*

$$\rho(X) = \sqrt{\mathbb{E}(X^2|X > \pi_{0.95}(X))}$$

Give a proof or a counterexample for each property.

[we can alternatively express this as $\rho(X) = \sqrt{\mathrm{TVaR}_{0.95}((X_+)^2)}$.]

Sub-additivity We first need to show that for positive random variables A and B, $\mathrm{TVaR}_{0.95}(AB) \leqslant \sqrt{\mathrm{TVaR}_{0.95}(A^2)\,\mathrm{TVaR}_{0.95}(B^2)}$ We have

$$\mathrm{TVaR}_{0.95}(AB) = \mathbb{E}(AB|AB > \pi_{0.95}) \leqslant \sqrt{\mathbb{E}(A^2|AB > \pi_{0.95}(AB))\mathbb{E}(B^2|AB > \pi_{0.95}(AB))} \leqslant \sqrt{\mathbb{E}(A^2$$

This gives

$$\mathrm{TVaR}_{0.95}((X_+Y_+)^2) \leqslant \mathrm{TVaR}_{0.95}((X_+)^2)\,\mathrm{TVaR}_{0.95}((Y_+)^2)$$

For random variables X and Y, we know that TVaR is coherent, and $((X+Y)_+)^2 \leqslant (X_+)^2 + (Y_+)^2 + 2(X_+Y_+)$ so we have

$$\rho(X+Y) = \sqrt{\mathrm{TVaR}_{0.95}(((X+Y)_+)^2)} \leqslant \sqrt{\mathrm{TVaR}_{0.95}(((X_+)^2 + (Y_+)^2 + 2(X_+Y_+)))} \leqslant \sqrt{\mathrm{TVaR}_{0.95}((}$$

Monotonicity This follows directly from monotonicity of TVaR — if $X \leqslant Y$, then $(X_+)^2 \leqslant (Y_+)^2$, so

Positive homogeneity For any $c > 0$, we have $((cX)_+)^2 = c^2((X_+)^2)$, so since TVaR satisfies positive homogeneity, we get

$$\rho(cX) = \sqrt{\mathrm{TVaR}_{0.95}(((cX)_+)^2)} = \sqrt{c^2\,\mathrm{TVaR}_{0.95}((X_+)^2)} = c\rho(X)$$

3. *Calculate the TVaR at the 99% level of a distribution with survival function* $S_X(x) = e^{\sqrt{x+1}-1-x}$ *for* $x > 0$.

The VaR at the 99% level is the solution to $S_X(x) = 0.01$, which is

$$\begin{aligned}
e^{\sqrt{x+1}-1-x} &= 0.01\\
1 + x - \sqrt{x+1} &= \log(100)\\
\sqrt{x+1} &= \frac{1+\sqrt{1+4\log(100)}}{2}\\
&= 2.70344507216
\end{aligned}$$

[The second solution of the quadratic equation is negative.] This gives $x = 2.70344507216^2 - 1 = 6.30861525819$

The TVaR is therefore

$$\begin{aligned}
6.30861525819 + \frac{1}{0.01}\int_{6.30861525819}^{\infty} S(x)\,dx &= 6.30861525819 + \frac{1}{0.01}\int_{6.30861525819}^{\infty} e^{\sqrt{x+1}-1-x}\,dx\\
&= 6.30861525819 + 100\int_{2.70344507216}^{\infty} 2ue^{u-u^2}\,du\\
&= 6.30861525819 + 100e^{0.25}\int_{2.70344507216}^{\infty} 2ue^{-(u-0.5)^2}\,du\\
&= 6.30861525819 + 100e^{0.25}\int_{2.70344507216}^{\infty} (2(u-0.5)+1)e^{-(u-0.5)^2}\,du\\
&= 6.30861525819 + 100e^{0.25}\int_{2.20344507216}^{\infty} (2v+1)e^{-v^2}\,dv\\
&= 6.30861525819 + 100e^{0.25}\left(\left[-e^{-v^2}\right]_{2.20344507216}^{\infty} + \int_{2.20344507216}^{\infty} e^{-v^2}\,dv\right)\\
&= 6.30861525819 + 100e^{0.25}\left(e^{-2.20344507216^2} + \sqrt{\pi}\Phi(-2.20344507216\sqrt{2})\right)\\
&= 7.517124
\end{aligned}$$

If calculating this integral analytically is too challenging, we can alternatively compute it numerically.

```
nstep<-50000000 # number of steps
stepsize<-0.000001

VaR<-6.30861525819
x<-VaR+seq_len(nstep)*stepsize # Steps of 0.000001
Sx<-exp(sqrt(x+1)-x-1)
TVaR<-100*sum(Sx)*stepsize+VaR
TVaR
```

This gives the same answer to 6 decimal places. Using fewer steps or a larger step size may produce less accurate answers.

4. *Which of the following distribution functions with parameters α, and β are scale distributions? Which have scale parameters?*

   (i) $F(x) = e^{-\beta e^{-x+\alpha}}$

   (ii) $F(x) = \frac{\frac{x}{\beta}+e^{\frac{x}{\alpha}}-1}{e^{\frac{x}{\alpha}}}$

   (iii) $F(x) = 1+\frac{\beta}{\alpha}-\frac{\beta}{x+\alpha}+e^{-\frac{\beta}{x+\beta}}$

   (i) is not a scale distribution since

   $$F_{cX}(x) = F\left(\frac{x}{c}\right) = e^{-\beta e^{-\frac{x}{c}+\alpha}}$$

   which is not of the same form.

   (ii) This is a scale distribution since

   $$F_{cX}(x) = F\left(\frac{x}{c}\right) = \frac{\frac{x}{c\beta}+e^{\frac{x}{c\alpha}}-1}{e^{\frac{x}{c\alpha}}}$$

   which is clearly of the same form with α replaced by $c\alpha$ and β replaced by $c\beta$. We also see that there is no scale parameter.

   (iii) We see that

   $$F_{cX}(x) = F\left(\frac{x}{c}\right) = 1+\frac{\beta}{\alpha}-\frac{\beta}{\frac{x}{c}+\alpha}+e^{-\frac{\beta}{\frac{x}{c}+\beta}=1+\frac{\beta}{\alpha}-\frac{c\beta}{x+c\alpha}+e^{-\frac{c\beta}{x+c\beta}}}$$

   which is clearly of the same form with α replaced by $c\alpha$ and β replaced by $c\beta$. We also see that there is no scale parameter.

5. *An insurance company observes the following sample of claims (in thousands):*

   0.3 0.4 1.0 1.3 1.6 2.6 7.2 10.3

   They use a kernel density model with Gaussian kernel with standard deviation 1. What is the variance of the fitted distribution?

   There are 8 sample points, The fitted distribution is a mixture of normal distributions. We can calculate the variance using law of total variance

   $$\mathrm{Var}(X) = \mathbb{E}\,\mathrm{Var}(X|Z) + \mathrm{Var}(\mathbb{E}(X|Z) = 1 + \mathrm{Var}(Z)$$

   where Z is the empirical distribution from the sample.

   We calculate $\mathbb{E}(Z) = \frac{0.3+0.4+1.0+1.3+1.6+2.6+7.2+10.3}{8} = 3.0875$ and $\mathbb{E}(Z^2) = \frac{0.09+0.16+1.0+1.69+2.56+6.76+51.84+106.09}{8} = 21.27375$. Thus, $\mathrm{Var}(Z) = 21.27375 - 3.0875^2 = 11.74109375$. This gives $\mathrm{Var}(X) = 12.74109375$.

Standard Questions

6. *An generalised Pareto distribution with* $\alpha = \tau$ *and* $\theta = 1$ *has mean* $\frac{\alpha}{\alpha-1}$ *and variance* $\frac{\alpha(2\alpha-1)}{(\alpha-1)^2(\alpha-2)}$. *You can simulate* n *random variables following this generalised Pareto distribution with the command*

```
sim=1/rbeta(n,shape1=alpha,shape2=alpha)
```

[This is simulating a beta distribution then taking the inverse.]

Based on the central limit theorem, if we take the average of a sample of n *generalised Pareto random variables, this should approximately follow a normal distribution with mean* $\frac{\alpha}{\alpha-1}$ *and variance* $\frac{\alpha(2\alpha-1)}{n(\alpha-1)^2(\alpha-2)}$. *Plot the distribution of this sample average for* $\alpha = 10$, $\alpha = 2.5$ *and* $\alpha = 2.05$, *for sample sizes 500, 1000, and 5000, and compare it with the normal distribution. What happens if we run the simulation with* $\alpha = 1.5$*?*

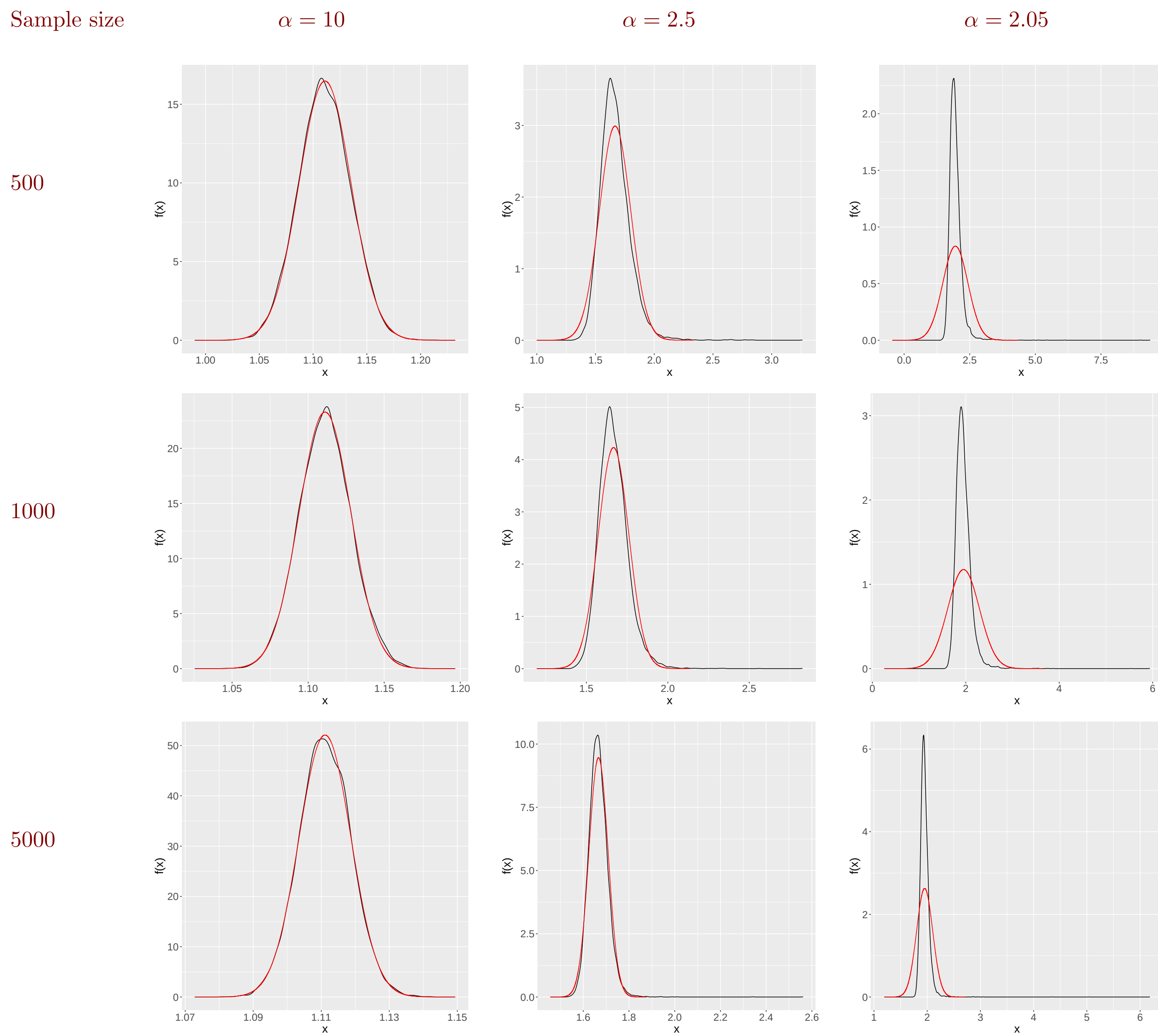
There is a typo' in the question — the correct code for simulating the data should be

```
sim=1/rbeta(n,shape1=alpha,shape2=alpha)-1
```

This is why in many of your plots, the distributions will not have lined up well.

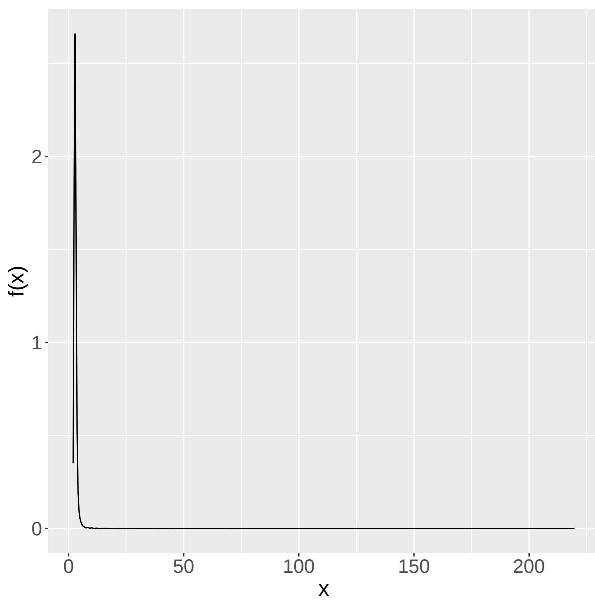
We run the simulations using the following code

```
library(ggplot2)

GenParCLTplot<-function(alpha,n,nsamp){
### alpha is the inverse gamma shape parameter
### n is the sample size
### m is the number of samples

    samp<-1/rbeta(n*nsamp,shape1=alpha,shape2=alpha)-1
    ## simulate generalised Pareto random variables

    samples<-matrix(samp,n,nsamp)
    means<-colMeans(samples)
    ## arranging into a matrix and using the column means function is
    ## an efficient way to calculate the sample means. You could also
    ## use a loop.

    if(alpha>2){
        dm<-alpha/(alpha-1)
        dv<-alpha*(2*alpha-1)/(alpha-1)^2/(alpha-2)

        x<-seq_len(100000)*0.0001*sqrt(dv/n)+dm-5*sqrt(dv/n)
    ## x covers 5 standard deviations either side of the mean
        ncomp<-geom_line(data=data.frame(x=x,y=dnorm(x-dm,sd=sqrt(dv/n))),
                  mapping=aes(x=x,y=y),
                  colour="red")
    }else{
        ncomp<-NULL
    }

    return(
        ggplot(data=data.frame(x=means),mapping=aes(x=x))+
        geom_density()+
        ncomp+
        scale_y_continuous(name="f(x)")+
        theme(axis.title=element_text(size=18),
            axis.text=element_text(size=16),
            plot.title=element_text(size=18,hjust=0.5))
    )

}


for(alpha in c(10,2.5,2.05)){
    for(ss in c(500,1000,5000)){
        pdf(paste("alpha",alpha,"ssize",ss,".pdf",sep=""))
        print(GenParCLTplot(alpha,ss,10000))
        dev.off()
    }
}
```

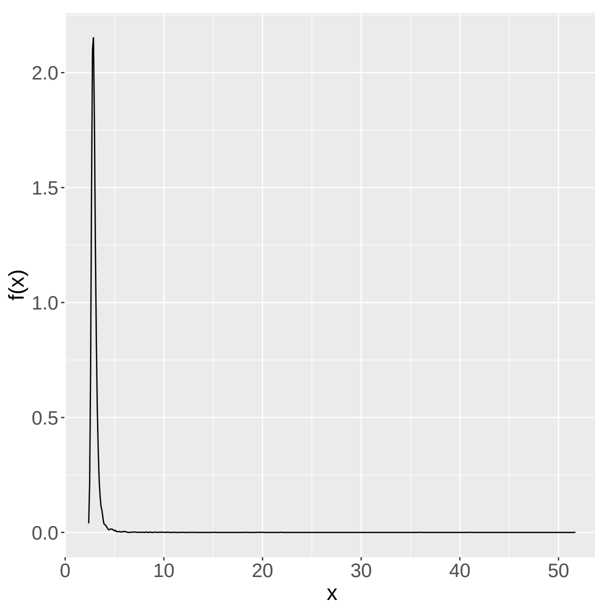


When $\alpha = 1.5$ the variance is infinite, and the sample mean will not converge to a distribution. As the sample size gets larger, the sample means get more spread out.

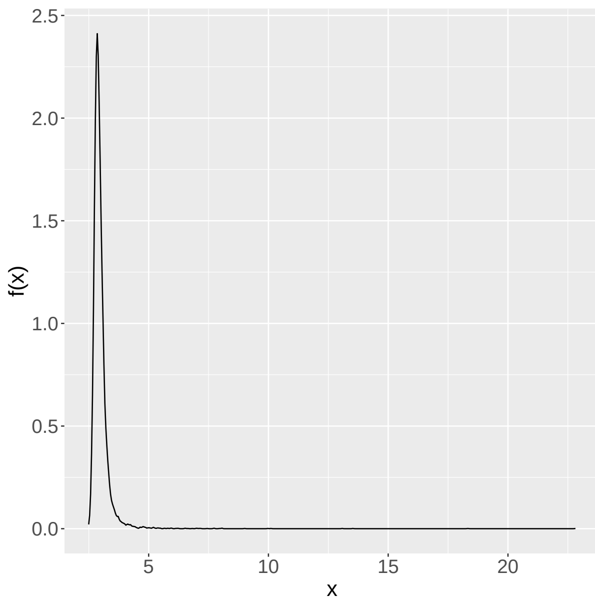
Sample size 1000:



Sample size 5000:



Sample size 10000:


2.5
2.0
1.5
1.0
0.5
0.0
f(x)
5
10
15
20
x