ACSC/STAT 3703, Actuarial Models I

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Homework Sheet 5

Model Solutions

Basic Questions

1. *The time T to claim settlement, in years, has density function*

$$f_T(t) = \begin{cases} \frac{Ce^{-\frac{t^2}{2}}}{(t+1)^2} & \text{if } 0 < t \leqslant 1 \\ \frac{Ce^{-e^{t-1}-\frac{1}{2}}}{4t^3} & \text{if } t > 1 \end{cases}$$

for some constant C. What is the density function for the time in days until claim settlement? [Assume all years are 365 days long.]

The density function is

$$f_{365T}(x)\frac{1}{365}f_T\left(\frac{t}{365}\right) = \begin{cases} \frac{365Ce^{-\frac{t^2}{2\times 365^2}}}{(t+365)^2} & \text{if } 0 < t \leqslant 365 \\ \frac{365^2Ce^{-e^{\frac{t-365}{365}}-\frac{1}{2}}}{4t^3} & \text{if } t > 365 \end{cases}$$

2. *Calculate the density function of X^{-2} when X follows a Pareto distribution with $\alpha = 2$ and $\theta = 5$.*

The density function of X is $f_X(x) = \frac{2\theta^2}{(\theta+x)^3}$, so the distribution function of X^{-2} is

$$f_{X^{-2}}(x) = \frac{1}{2}x^{-\frac{3}{2}}f_X(x^{-\frac{1}{2}}) = \frac{1}{2}x^{-\frac{3}{2}}\frac{2\theta^2}{(\theta+x^{-\frac{1}{2}})^3} = \frac{\theta^2}{(\theta\sqrt{x}+1)^3} = \frac{25}{(5\sqrt{x}+1)^3}$$

3. *The time in years until a claim is processed is a random variable T with moment generating function $M_T(t) = te^t$. Inflation is at an annual rate of 4%. What is the skewness of inflation during the processing period of a random claim?*

Let I be the inflation. We have that $I = (1.04)^T$. Thus, the raw moments of I are

$$\mu = \mathbb{E}((1.04)^T) = M_T(\log(1.04)) = \log(1.04)e^{\log(1.04)} = 1.04\log(1.04)$$
$$\mu_2' = \mathbb{E}((1.04)^{2T}) = M_T(2\log(1.04)) = 2\log(1.04)e^{2\log(1.04)} = 2(1.04)^2\log(1.04)$$
$$\mu_3' = \mathbb{E}((1.04)^{3T}) = M_T(3\log(1.04)) = 3\log(1.04)e^{3\log(1.04)} = 3(1.04)^3\log(1.04)$$

The centred moments are therefore:

$$\mu_2 = \mu_2' - \mu^2 = 2(1.04)^2 \log(1.04) - (1.04 \log(1.04))^2 = 1.04^2(2\log(1.04) - \log(1.04)^2) = 0.0831784599829$$
$$\mu_3 = \mu_3' - 3\mu\mu_2' + 2\mu^3 = (1.04)^3(3\log(1.04) - 6\log(1.04)^2 + 3\log(1.04)^3) = 0.122175471061$$

The skewness is

$$\frac{\mu_3}{\mu_2^{\frac{3}{2}}} = \frac{0.122175471061}{0.0831784599829^{\frac{3}{2}}} = 5.09293002546$$

4. *X is a mixture of 2 distributions:*

   - *With probability p, X follows a Pareto distribution with $\alpha = 2.2$ and $\theta = 48$.*
   - *With probability $1-p$, X follows a Weibull distribution with $\tau = 0.5$ and $\theta = 36$.*

   The variance of X is 23216. What is the probability that X is more than 400?

   The variance of X is

   $$48^2 \frac{2.2}{1.2^2 \times 0.2} p + 36^2 \left(\Gamma(5) - \Gamma(3)^2\right)(1-p) + \left(\frac{48}{1.2} - 36\Gamma(3)\right)^2 p(1-p) = 17600p + 25920(1-p) + 1024p(1-p)$$

   We therefore solve

   $$\begin{aligned} 17600p + 25920(1-p) + 1024p(1-p) &= 23216 \\ 1100p + 1620(1-p) + 64p(1-p) &= 1451 \\ 169 &= 456p + 64p^2 \\ p &= \frac{\sqrt{456^2 + 4 \times 64 \times 169} - 456}{128} \\ &= 0.353113776922 \end{aligned}$$

   The probability that X exceeds 400 is therefore

   $$0.353113776922\left(\frac{48}{448}\right)^{2.2} + 0.646886223078 e^{-\sqrt{\frac{400}{36}}} = 0.0256702013277$$

5. *For a particular claim, an insurance company has observed the following claim sizes:*

   1.1 1.5 1.8 2.2 2.9 4.3 7.0 11.4

They use a kernel smoothing model with a uniform kernel. They choose the bandwidth so that the variance of the kernel smoothing model is 17. What is the probability for this model that a random claim will exceed 7.3?

First, we need to find the bandwidth used. The variance of a uniform distribution with width $2b$ is $\frac{b^2}{3}$, and the variance of the empirical distribution is

$$\frac{1.1^2+1.5^2+1.8^2+2.2^2+2.9^2+4.3^2+7.0^2+11.4^2}{8}-\left(\frac{1.1+1.5+1.8+2.2+2.9+4.3+7.0+11.4}{8}\right)^2=10.974375$$

We therefore get $10.974375+\frac{b^2}{3}=17$, which gives $b=4.25169084013$ The probability that a random claim exceeds 7.3 is therefore

$$\frac{1}{8\times 2\times 4.25169084013}(4.3+4.25169084013-7.3+7.0+4.25169084013-7.3+11.4+4.25169084013-7.3)=0.199260027218$$

Standard Questions

6. *An insurance company models the claims of an individual (in dollars) as following a Pareto distribution with $\theta = 100$ and α varying between individuals. For a random individual, $\alpha - 1$ is assumed to follow a gamma distribution with shape parameter $\alpha = 2$ and scale parameter θ. The expected claim amount is \$3,600. What is the probability that a claim exceeds \$100,000?*

   For the Pareto distribution with $\theta = 100$, the expected claim amount is $\frac{100}{\alpha-1}$, so the overall expected claim is

   $$\mathbb{E}\left(\frac{100}{\alpha-1}\right)=100\int_0^\infty \frac{1}{a}\frac{ae^{-\frac{a}{\theta}}}{\theta^2}\,da=100\theta^{-2}\left[-\theta e^{-\frac{a}{\theta}}\right]_0^\infty=100\theta^{-1}$$

   Thus we have $\theta = \frac{1}{36}$ for the gamma distribution.

   For a given value of α, the probability that a claim exceeds \$100,000 is $\left(\frac{100}{100000+100}\right)^\alpha = 1001^{-\alpha}$. Thus, the probability is

   $$\begin{aligned}\mathbb{E}\left(1001^{-\alpha}\right)&=36^2\int_0^\infty 1001^{-1-a}ae^{-36a}\,da\\&=\frac{36^2}{1001}\int_0^\infty ae^{-(36+\log(1001))a}\,da\\&=\frac{36^2}{1001(36+\log(1001))^2}\\&=0.000703200387282\end{aligned}$$

7. *An insurance company models claims X as following the exponential of a distribution with moment generating function $M(t) = \frac{e^{3t}}{(t+1)}$. [A typo in the original question gave this as $M(t) = (t+1)e^{3t}$, which is not the M.G.F. of a distribution.] They want to transform the distribution by raising to a power. To what power should they raise the distribution in order for the skewness to exist and be equal to 5?*

   (i) 4.3252

   (ii) 6.9924

   (iii) 8.5905

   (iv) 11.9331

   Justify your answer.

   Using the M.G.F. with the typo:

   Since $X = e^Z$ where Z has moment generating function $M(t) = (t+1)e^{3t}$, we have $\mathbb{E}(X^\alpha) = \mathbb{E}(e^{\alpha Z}) = M(\alpha)$. In particular, the first three raw moments of X^τ are

$$\begin{aligned}\mu &= (\tau+1)e^{3\tau}\\ \mu_2' &= (2\tau+1)e^{6\tau}\\ \mu_3' &= (3\tau+1)e^{9\tau}\end{aligned}$$

   The central moments are therefore

$$\begin{aligned}\mu_2 &= (2\tau+1)e^{6\tau} - \left((\tau+1)e^{3\tau}\right)^2\\ &= -2\tau^2 e^{6\tau}\end{aligned}$$

   Since this is not positive, we see that this cannot be the correct moment generating function.

   Using the intended M.G.F. :

   Since $X = e^Z$ where Z has moment generating function $M(t) = \frac{e^{3t}}{t+1}$, we have $\mathbb{E}(X^\alpha) = \mathbb{E}(e^{\alpha Z}) = M(\alpha)$. In particular, the first three raw moments of X^τ are

$$\begin{aligned}\mu &= \frac{e^{3\tau}}{\tau+1}\\ \mu_2' &= \frac{e^{6\tau}}{2\tau+1}\\ \mu_3' &= \frac{e^{9\tau}}{3\tau+1}\end{aligned}$$

The central moments are therefore

$$\begin{aligned}
\mu_2 &= \frac{e^{6\tau}}{2\tau+1} - \left(\frac{e^{3\tau}}{\tau+1}\right)^2 \\
&= e^{6\tau}\left(\frac{1}{2\tau+1} - \frac{1}{(\tau+1)^2}\right) \\
&= e^{6\tau}\left(\frac{\tau^2}{(\tau+1)^2(2\tau+1)}\right) \\
\mu_3 &= \frac{e^{9\tau}}{3\tau+1} - 3\frac{e^{6\tau}}{2\tau+1}\frac{e^{3\tau}}{\tau+1} + 2\left(\frac{e^{3\tau}}{\tau+1}\right)^3 \\
&= e^{9\tau}\left(\frac{1}{3\tau+1} - \frac{3}{(\tau+1)(2\tau+1)} + \frac{2}{(\tau+1)^3}\right) \\
&= e^{9\tau}\frac{(\tau+1)^3(2\tau+1) - 3(\tau+1)^2(3\tau+1) + 2(2\tau+1)(3\tau+1)}{(\tau+1)^3(2\tau+1)(3\tau+1)} \\
&= e^{9\tau}\frac{2\tau^4 + 7\tau^3 + 9\tau^2 + 5\tau + 1 - 3(3\tau^3 + 7\tau^2 + 5\tau + 1) + 2(6\tau^2 + 5\tau + 1)}{(\tau+1)^3(2\tau+1)(3\tau+1)} \\
&= e^{9\tau}\frac{2\tau^3(\tau-1)}{(\tau+1)^3(2\tau+1)(3\tau+1)}
\end{aligned}$$

The skewness is then

$$\frac{\left(\frac{2\tau^3(\tau-1)}{(\tau+1)^3(2\tau+1)(3\tau+1)}\right)}{\left(\frac{\tau^2}{(\tau+1)^2(2\tau+1)}\right)^{\frac{3}{2}}} = \frac{2\tau^3(\tau-1)\sqrt{(2\tau+1)}}{\tau^3(3\tau+1)} = \frac{2(\tau-1)\sqrt{(2\tau+1)}}{3\tau+1}$$

We set this equal to 5 and solve:

$$\begin{aligned}
\frac{2(\tau-1)\sqrt{(2\tau+1)}}{3\tau+1} &= 5 \\
2(\tau-1)\sqrt{(2\tau+1)} &= 5(3\tau+1) \\
4(\tau-1)^2(2\tau+1) &= 25(3\tau+1)^2 \\
8\tau^3 - 12\tau^2 - 4 &= 225\tau^2 + 150\tau + 25 \\
8\tau^3 - 237\tau^2 - 150\tau - 29 &= 0
\end{aligned}$$

We try the given values

	τ	$8\tau^3 - 237\tau^2 - 150\tau - 29$
(i)	4.3252	−1055.548
(ii)	6.9924	0.013
(iii)	8.5905	1718.080
(iv)	11.9331	8946.970

We see that (ii) $\tau = 6.9924$ is the answer.