ACSC/STAT 3703, Actuarial Models I

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Homework Sheet 6

Model Solutions

Basic Questions

1. *Let X follow a negative binomial distribution with $r = 2.5$ and $\beta = 3.1$. What is the probability that $X = 5$?*

   The probability is

$$P(X = 5) = \binom{1.5 + 5}{5} \frac{1}{4.1^{2.5}} \left(\frac{3.1}{4.1}\right)^5 = 0.0851618153831$$

2. *The number of claims on each insurance policy over a given time period is observed as follows:*

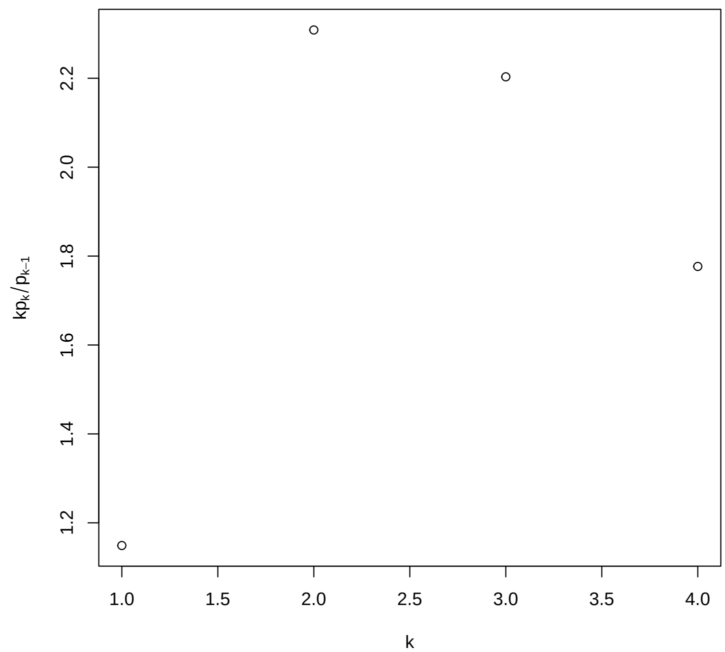
Number of claims	*Number of policies*
0	*423*
1	*486*
2	*561*
3	*412*
4	*183*
5 or more	*107*

   Which distribution(s) from the $(a, b, 0)$-class and $(a, b, 1)$-class appear most appropriate for modelling this data?

   We estimate

n	$n\frac{p_n}{p_{n-1}}$
1	$1\frac{486}{423} = 1.14893617021$
2	$2\frac{561}{486} = 2.30864197531$
3	$3\frac{412}{561} = 2.20320855615$
4	$4\frac{183}{412} = 1.77669902913$

   We can plot a graph of $n\frac{p_n}{p_{n-1}}$ against n. For a distribution from the $(a, b, 0)$ class, this should be linear with slope a and intercept b. For a distribution from the $(a, b, 1)$ class, all points for $n \neq 1$ should be linear.



On this graph, the first point is clearly not on the same line as the other three points, so the distribution should clearly be zero-modified. For the other three points, the slope is negtive, so $a < 0$. This suggests a zero-modified binomial distribution.

3. *X follows an extended modified negative binomial distribution with $r = -0.5$ and $\beta = 1.7$, and $p_0 = 0.6$. What is $P(X = 4)$?*

For the truncated ETNB with $r = -0.5$ and $\beta = 1.7$, we have $p_1 = \frac{r\beta}{(1+\beta)((1+\beta)^r-1)} = \frac{-0.5\times1.7}{2.7(2.7^{-0.5}-1)} = 0.804290309721$ We also have $a = \frac{\beta}{1+\beta} = \frac{1.7}{2.7} = 0.62962962963$ and $b = (r-1)a = -1.5\times0.62962962963 = -0.944444444445$.

$$p_2 = \left(0.62962962963 - \frac{0.944444444445}{2}\right) \times 0.804290309721 = 0.126601252456$$

$$p_3 = \left(0.62962962963 - \frac{0.944444444445}{3}\right) \times 0.126601252456 = 0.0398559498473$$

$$p_4 = \left(0.62962962963 - \frac{0.944444444445}{4}\right) \times 0.0398559498473 = 0.0156840543381$$

Now for the distribution with $p_0 = 0.6$, we have $P(X = 4) = 0.0156840543381\times 0.4 = 0.00627362173524$.

4. *Let X follow a mixed negative binomial distribution with $\beta = 0.5$ and r following a gamma distribution with $\alpha = 3$ and $\theta = 0.7$. What is the probability that $X = 3$?*

For a given value of r,

$$P(X=3) = 1.5^{-r}\binom{r+2}{3}3^{-3} = \frac{1.5^{-r}r(r+1)(r+2)}{162}$$

For the mixed distribution, $P(X = 3)$ is therefore given by the expectation

$$\mathbb{E}\left(\frac{1.5^{-R}(R(R+1)(R+2)}{162}\right)$$

For the gamma distribution, we compute the expectation:

$$\begin{aligned}
P(X=3) &= \int_0^\infty \frac{e^{-\log(1.5)r}r(r+1)(r+2)}{162}\frac{r^2e^{-\frac{r}{0.7}}}{2\times 0.7^3}\,dr \\
&= \int_0^\infty \frac{r^3(r+1)(r+2)e^{-r\left(\frac{1}{0.7}+\log(1.5)\right)}}{324\times 0.7^3}\,dr \\
&= \frac{\int_0^\infty r^5e^{-r\left(\frac{1}{0.7}+\log(1.5)\right)}\,dr + 3\int_0^\infty r^4e^{-r\left(\frac{1}{0.7}+\log(1.5)\right)}\,dr + 2\int_0^\infty r^5e^{-r\left(\frac{1}{0.7}+\log(1.5)\right)}\,dr}{111.132} \\
&= \frac{\frac{\Gamma(6)}{\left(\frac{1}{0.7}+\log(1.5)\right)^6} + 3\frac{\Gamma(5)}{\left(\frac{1}{0.7}+\log(1.5)\right)^5} + 2\frac{\Gamma(4)}{\left(\frac{1}{0.7}+\log(1.5)\right)^4}}{111.132} \\
&= 0.0691373111194
\end{aligned}$$

Standard Questions

5. *A random variable X is assumed to have distribution in the $(a, b, 1)$-class. The probability mass function satisfies the equations*

$$\begin{aligned}
P(X=5) &= 3P(X=3) \\
P(X=6) &= 2P(X=4)
\end{aligned}$$

What is the largest possible value of $P(X = 7)$?

The equations gives

$$3 = \frac{p_5}{p_3} = \frac{p_5}{p_4}\frac{p_4}{p_3} = \left(a+\frac{b}{5}\right)\left(a+\frac{b}{4}\right)$$

which we can rewrite as

$$b^2 + 9ab + 20a^2 = 60$$

and

$$2 = \frac{P(X=6)}{P(X=4)} = \frac{p_6}{p_5}\frac{p_5}{p_4} = \left(a + \frac{b}{6}\right)\left(a + \frac{b}{5}\right) = \frac{b^2 + 11ab + 30a^2}{30}$$

so we have

$$b^2 + 11ab + 30a^2 = 60$$

We subtract one equation from the other to get

$$2ab + 10a^2 = 0$$

so $a = 0$ or $b = -5a$. For $b = -5a$, we get $b^2 - 9ab + 20a^2 = 25a^2 - 45a^2 + 20a^2 = 0$, so there is no solution to the equations. For $a = 0$, we get $b = \sqrt{60}$, so the distribution is a zero-modified Poisson distribution with $\lambda = \sqrt{60}$. For this distribution, we have $P(X = 7) = \frac{e^{-\sqrt{60}}\sqrt{60}^7}{7!(1-e^{-\sqrt{60}})}(1 - p_0)$ which is clearly maximised for the zero-truncated distribution $p_0 = 0$. This gives $P(X = 7) = \frac{e^{-\sqrt{60}}\sqrt{60}^7}{7!(1-e^{-\sqrt{60}})} = 0.143633622646$.

6. *If we extend the* $(a, b, 0)$ *class to a class satisfying the recurrence* $p_n = \left(a + \frac{b}{n} + \frac{c}{n(n+1)}\right)p_{n-1}$*, what values of* a*,* b *and* c *give rise to valid discrete distributions?*

As in the $(a, b, 0)$ case, when $a \geqslant 1$, $\sum_{n=0}^{\infty} p_n = \infty$, unless $p_n = 0$ for all sufficiently large n. That is, we can have cases with $a \geqslant 1$, $b < 0$ and $c > 0$ if there is some k for which $a + \frac{b}{k} + \frac{c}{k(k+1)} = 0$. We also need $a + \frac{b}{n} + \frac{c}{n(n+1)} > 0$ for all $n < k$. By induction, it will be sufficient to ensure $\frac{b}{k-1} + \frac{c}{k(k-1)} > \frac{b}{k} + \frac{c}{k(k+1)}$. We have

$$\begin{aligned}\frac{b}{k-1} + \frac{c}{k(k-1)} - \left(\frac{b}{k} + \frac{c}{k(k+1)}\right) &= \frac{b}{k(k-1)} + \frac{2c}{(k-1)k(k+1)} \\ &= \frac{1}{k-1}\left(\frac{c}{k(k+1)} - a\right)\end{aligned}$$

so we need $c > ak(k+1)$ or equivalently $b < -2ak$. That is, for $a \geqslant 1$, and $\beta > 2a$, the triple $(a, -k\beta, k(k+1)(\beta - a))$ is valid.

For $a < 1$, the triple will always work provided the values are all positive. For $a \geqslant 0$, $b \geqslant 0$, this requires $c \geqslant -(a + b)$. For $a \geqslant 0$ and $b \leqslant 0$, the situation is more complicated. We need $a + \frac{b}{n} + \frac{c}{n(n+1)} > 0$ for all n. Equivalently, we need $an(n+1) + bn + c > 0$ for all n. Rewriting this as $n^2 + \left(\frac{b}{a} + 1\right)n + \frac{c}{a} > 0$ we observe

$$n^2 + \left(\frac{b}{a} + 1\right)n + \frac{c}{a} = \left(n + \left(\frac{b}{2a} + \frac{1}{2}\right)\right)^2 + \frac{c}{a} - \left(\frac{b}{2a} + \frac{1}{2}\right)^2$$

Thus, it is sufficient if $\frac{c}{a} \geqslant \left(\frac{b}{2a}+\frac{1}{2}\right)^2$. Since n is restricted to be an integer, there are a few other cases that can work. The expression is minimised when $n=-\frac{1}{2}-\frac{b}{2a}$. If this is not an integer, the nearest integer will work. Thus, we need $\frac{c}{a} \geqslant \left(\frac{b}{2a}+\frac{1}{2}\right)^2 - \left(\frac{1}{2}+\frac{b}{2a}-\lceil\frac{1}{2}+\frac{b}{2a}\rceil\right)^2$.

There is also the possibility that $p_n=0$ for some $n<-\frac{1}{2}-\frac{b}{2a}$, which will happen if $c=-(n+1)b-n(n+1)a$ for some $n<-\frac{1}{2}-\frac{b}{2a}$.

For $a<0$, for large enough n, we have $a+\frac{b}{n}+\frac{c}{n(n+1)}<0$, so this will only be possible if $a+\frac{b}{n}+\frac{c}{n(n+1)}=0$ for some n. That is $c=-n(n+1)a-(n+1)b$. Furthermore, we will need $an(n+1)+b(n+1)+c>0$ for all smaller n. Since $an(n+1)+b(n+1)+c$ is concave, this will work as long as it holds for $n=1$, That is $2a+2b+c>0$. Substituting $c=-n(n+1)a-(n+1)b$, this becomes $b<-\frac{(n^2+n-2)a}{n-1}$.

In summary, the following combinations give valid discrete distributions:

- $a \geqslant 1, b<-2ak, c=-k(k+1)a-(k+1)b$
- $0 \leqslant a<1, b \geqslant 0, c \geqslant -(a+b)$
- $0 \leqslant a<1, b<0, c \geqslant a\left(\left(\frac{b}{2a}+\frac{1}{2}\right)^2-\left(\frac{1}{2}+\frac{b}{2a}-\lceil\frac{1}{2}+\frac{b}{2a}\rceil\right)^2\right)$
- $0 \leqslant a<1, b<0, c=-(k+1)b-k(k+1)a$ for some $k<-\frac{1}{2}-\frac{b}{2a}$.
- $a<0, b<-\frac{(k^2+k-2)a}{k-1}, c=-(k+1)b-k(k+1)a$