ACSC/STAT 3703, Actuarial Models I

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Homework Sheet 7

Model Solutions

Basic Questions

1. *An insurance company has an insurance policy where the loss amount follows an inverse Gamma distribution with $\alpha = 3$ and $\theta = 200$. Calculate the expected payment per claim if the company introduces a deductible of d.*

For the inverse Gamma distribution $f(x) = \frac{200^3 x^{-4} e^{-\frac{400}{x}}}{2}$. The expected payment per loss is

$$\begin{aligned}
\int_d^\infty \frac{200^3(x-d)x^{-4}e^{-\frac{200}{x}}}{2}\,dx &= \int_d^\infty \frac{200^3 x^{-3}e^{-\frac{200}{x}}}{2}\,dx - d\int_d^\infty \frac{200^3 x^{-4}e^{-\frac{200}{x}}}{2}\,dx \\
&= \int_d^\infty 200^2\frac{x^{-1}}{2}\left(200x^{-2}e^{-\frac{200}{x}}\right)dx - d\int_d^\infty \frac{200^2x^{-2}}{2}\left(200x^{-2}e^{-\frac{200}{x}}\right)dx \\
&= \left(\left[200^2\frac{x^{-1}}{2}e^{-\frac{200}{x}}\right]_d^\infty + \int_d^\infty 100\left(200x^{-2}e^{-\frac{200}{x}}\right)dx\right) \\
&\qquad - d\left(\left[\frac{200^2x^{-2}}{2}e^{-\frac{200}{x}}\right]_d^\infty + \int_d^\infty 200x^{-1}(200x^{-2}e^{-\frac{200}{x}})\,dx\right) \\
&= \left(200^2\frac{-e^{-\frac{200}{d}}}{2d} + 100\left(1-e^{-\frac{200}{d}}\right)\right) - d\left(\frac{-200^2e^{-\frac{200}{d}}}{2d^2} + \left(\frac{-200e^{-\frac{200}{d}}}{d} + 1 - e^{-\frac{200}{d}}\right)\right) \\
&= 100 - d + e^{-\frac{200}{d}}(100+d)
\end{aligned}$$

The probability that a loss results in a claim is

$$\begin{aligned}
\int_d^\infty \frac{200^3x^{-4}e^{-\frac{200}{x}}}{2}\,dx &= \frac{-200^2e^{-\frac{200}{d}}}{2d^2} + \left(\frac{-200e^{-\frac{200}{d}}}{d} + 1 - e^{-\frac{200}{d}}\right) \\
&= 1 - e^{-\frac{200}{d}}\left(1 + \frac{200}{d} + \frac{20000}{d^2}\right)
\end{aligned}$$

Thus, the expected payment per claim is

$$\frac{100 - d + e^{-\frac{200}{d}}(100+d)}{1 - e^{-\frac{200}{d}}\left(1 + \frac{200}{d} + \frac{20000}{d^2}\right)} = \frac{d^2(100-d) + e^{-\frac{200}{d}}d^2(100+d)}{d^2 - e^{-\frac{200}{d}}(d^2 + 200d + 20000)}$$

2. *The severity of a loss on a home insurance policy follows a log-logistic distribution with $\gamma = 2$ and $\theta = 1500$. Calculate the loss eliminatrion ratio of a deductible of \$2,000.*

   Without the deductible, the expected payment per loss is $1500\Gamma\left(1+\frac{1}{2}\right)\Gamma\left(1-\frac{1}{2}\right) = 750\pi$. With the deductible, the expected payment is

$$\begin{aligned}\int_{2000}^{\infty} \frac{1500^2}{1500^2+x^2}\,dx &= 1500\int_{\frac{4}{3}}^{\infty} \frac{1}{1+u^2}\,du\\ &= 1500\left[\tan^{-1}(u)\right]_{\frac{4}{3}}^{\infty}\\ &= 1500\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{4}{3}\right)\right)\end{aligned}$$

   Therefore the loss elimination ratio is

$$1-\frac{1500\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{4}{3}\right)\right)}{750\pi} = \frac{2\tan^{-1}\left(\frac{4}{3}\right)}{\pi} = 59.03\%$$

3. *An insurance company has a policy where losses follow a Pareto distribution with $\alpha = \frac{1}{3}$ and $\theta = 1000$. The company wants the TVaR at the 95% level for this policy to be \$10,000,000. What policy limit should the company put on the policy to achieve this?*

   The survival function of the Pareto distribution is $S(x) = \left(\frac{1000}{1000+x}\right)^{\frac{1}{3}}$. The VaR at the 95% level is therefore obtained by solving

$$\begin{aligned}\left(\frac{1000}{1000+x}\right)^{\frac{1}{3}} &= 0.05\\ \frac{1000+x}{1000} &= 8000\\ x &= 7999000\end{aligned}$$

   With limit u, the TVaR is

$$\begin{aligned}\mathrm{TVaR}_{0.95}(X) &= 7999000 + 20\int_{7999000}^{u} S(x)\,dx\\ &= 7999000 + 20\int_{7999000}^{u} \left(\frac{1000}{1000+x}\right)^{\frac{1}{3}}\,dx\\ &= 7999000 + 20\int_{8000000}^{u+1000} 1000^{\frac{1}{3}} v^{-\frac{1}{3}}\,dv\\ &= 7999000 + 200\left[\frac{3}{2}v^{\frac{2}{3}}\right]_{8000000}^{u+1000}\\ &= 7999000 + 300\left((u+1000)^{\frac{2}{3}} - 40000\right)\end{aligned}$$

where we have used the substitution $v = 1000 + x$. We therefore need to solve

$$\begin{aligned}
7999000 + 300\left((u+1000)^{\frac{2}{3}} - 40000\right) &= 10000000\\
300\left((u+1000)^{\frac{2}{3}} - 40000\right) &= 2001000\\
\left((u+1000)^{\frac{2}{3}} - 40000\right) &= 6670\\
(u+1000)^{\frac{2}{3}} &= 46670\\
u + 1000 &= 46670^{\frac{3}{2}} = 10082232.3403\\
u &= 10083232.3403
\end{aligned}$$

4. *Aggregate payments have a compound distribution. The frequency distribution is negative binomial with $r = 2.4$ and $\beta = 0.5$. The severity distribution has mean 2,278 and variance 11,925,000. Use a Pareto approximation to aggregate payments to estimate the expected payment on a reinsurance policy with attachment point $500,000.*

   The frequency distribution has mean $2.4 \times 0.4 = 1.2$ and variance $2.4 \times 0.5 \times 1.5 = 1.8$. Therefore the aggregate loss distribution has mean $1.2 \times 2278 = 2733.6$ and variance $1.2 \times 11925000 + 1.8 \times 2278^2 = 23650711.2$. Setting these equal to the mean and variance of a Pareto distribution with parameters α and θ gives

$$\begin{aligned}
\frac{\theta}{\alpha - 1} &= 2733.6\\
\frac{\alpha\theta}{(\alpha-1)^2(\alpha-2)} &= 23650711.2\\
\frac{\alpha}{\alpha-2} &= \frac{23650711.2}{2733.6^2} = 3.16500407378\\
1 - \frac{2}{\alpha} &= 0.315955359516\\
\alpha &= 2.92378579062\\
\theta &= 2733.6 \times 1.92378579062 = 5258.86083724
\end{aligned}$$

   For these parameters, the expected payment on a reinsurance policy with attachment point $500,000 is

$$\int_{500000}^{\infty}\left(\frac{5258.86083724}{5258.86083724+x}\right)^{2.92378579062} dx = 5258.86083724^{2.92378579062}\int_{505258.86083724}^{\infty} u^{-2.92378579062}\,du$$
$$= 5258.86083724^{2.92378579062}\left[-\frac{u^{-1.92378579062}}{1.92378579062}\right]_{505258.86083724}^{\infty}$$
$$= \frac{5258.86083724^{2.92378579062}}{1.92378579062(505258.86083724)^{1.92378579062}} = 0.419367174528$$

Standard Questions

5. *For a certain insurance policy, losses follow a Pareto distribution. with no policy limit, a deductible of $1,000 would achieve a loss elimination ratio of 10%, and a deductible of $5,000 would achieve a loss elimination ratio of 20%. What is the loss elimination ratio of a $1,000 deductible with a policy limit of $100,000 applied after the deductible.*

   [The parameter θ for the Pareto distribution is one of the following values:

   (i) $\theta = 437.04846$

   (ii) $\theta = 630.39300$

   (iii) $\theta = 883.47821$

   (iv) $\theta = 1522.03242$

   ]

   We first need to find the values of α and θ. With deductible d and limit $u-d$, the expected payment per loss is

$$\int_0^u S(x)\,dx = \int_d^u \left(\frac{\theta}{\theta+x}\right)^{\alpha} dx$$
$$= \int_{d+\theta}^{u+\theta} \theta^{\alpha} v^{-\alpha}\,dv$$
$$= \theta^{\alpha}\left[-\frac{v^{1-\alpha}}{\alpha-1}\right]_{d+\theta}^{u+\theta}$$
$$= \frac{\theta^{\alpha}}{\alpha-1}\left(\frac{1}{(d+\theta)^{\alpha-1}} - \frac{1}{(u+\theta)^{\alpha-1}}\right)$$

   With no policy limit, The loss elimination ratio for a deductible of d is therefore

$$1 - \frac{\left(\frac{1}{(d+\theta)^{\alpha-1}}\right)}{\left(\frac{1}{(\theta)^{\alpha-1}}\right)} = 1 - \left(\frac{\theta}{d+\theta}\right)^{\alpha-1}$$

Substituting the given values for $d = \$1,000$ and $d = \$5,000$, we get the equations

$$\left(\frac{\theta}{1000+\theta}\right)^{\alpha-1} = 0.9$$
$$\left(\frac{\theta}{5000+\theta}\right)^{\alpha-1} = 0.8$$
$$(\alpha-1)\log\left(\frac{\theta}{1000+\theta}\right) = \log(0.9)$$
$$(\alpha-1)\log\left(\frac{\theta}{5000+\theta}\right) = \log(0.8)$$
$$\frac{\log\left(\frac{\theta}{5000+\theta}\right)}{\log\left(\frac{\theta}{1000+\theta}\right)} = \frac{\log(0.8)}{\log(0.9)}$$
$$\frac{\log(\theta)-\log(5000+\theta)}{\log(\theta)-\log(1000+\theta)} = 2.1179048899$$

We compute $\frac{\log(\theta)-\log(5000+\theta)}{\log(\theta)-\log(1000+\theta)} - 2.1179048899$ for the given values of θ:

θ	$\frac{\log(\theta)-\log(5000+\theta)}{\log(\theta)-\log(1000+\theta)} - 2.1179048899$
437.0485	5.044288×10^{-9}
630.3930	0.1863628
883.4782	0.3867380
1522.0324	0.7634537

We see that $\theta = 437.04846$. We solve for the corresponding value of α:

$$\left(\frac{\theta}{1000+\theta}\right)^{\alpha-1} = 0.9$$
$$\alpha - 1 = \frac{\log(0.9)}{\log\left(\frac{437.04846}{1437.04846}\right)}$$
$$\alpha = 1.08851574558$$

With a policy limit of \$100,000 applied after the deductible, the expected payment on a policy with no deductible is

$$\frac{\theta^\alpha}{\alpha-1}\left(\frac{1}{(\theta)^{\alpha-1}} - \frac{1}{(100000+\theta)^{\alpha-1}}\right) = \frac{437.05^{1.0885}}{1.0885-1}\left(\frac{1}{(437.05)^{1.0885-1}} - \frac{1}{(100437.05)^{1.0885-1}}\right)$$
$$= 1886.17845816$$

With a deductible of \$1,000, the expected payment is

$$\frac{\theta^\alpha}{\alpha-1}\left(\frac{1}{(1000+\theta)^{\alpha-1}}-\frac{1}{(101000+\theta)^{\alpha-1}}\right)=\frac{437.05^{1.0885}}{1.0885-1}\left(\frac{1}{(1437.05)^{1.0885-1}}-\frac{1}{(101437.05)^{1.0885-1}}\right)$$
$$=1395.10093148$$

so the loss elimination ratio is $1-\frac{1395.10093148}{1886.17845816}=26.04\%$.

6. *An insurance company models loss frequency as negative binomial with $r=0.2$ and $\beta=160$, and loss severity as Pareto with $\alpha=0.5$ and $\theta=1600$. The insurer sets a policy limit u per loss. The insurer buys stop-loss reinsurance for aggregate losses above the expected aggregate losses, the price for which is based on using a Pareto distribution for aggregate losses with parameters fitted using the method of moments. The expected payment on this reinsurance is $500,000. What is the policy limit u?*

   The negative binomial distribution has mean $0.2\times160=32$ and variance $0.2\times160\times161=5152$. The expected payment on the Pareto distribution with policy limit u is

$$\begin{aligned}\int_0^u\sqrt{\frac{1600}{1600+x}}\,dx&=40\int_{1600}^{1600+u}v^{-\frac{1}{2}}\,dv\\&=40\left[2\sqrt{v}\right]_{1600}^{1600+u}\\&=80\left(\sqrt{u+1600}-40\right)\end{aligned}$$

   To simplify the algebra, we let $s=\sqrt{u+1600}$. The expected square of the payment with policy limit u is

$$\begin{aligned}\int_0^u 2x\sqrt{\frac{1600}{1600+x}}\,dx&=80\int_{1600}^{u+1600}(v-1600)v^{-\frac{1}{2}}\,dv\\&=80\left(\int_{1600}^{u+1600}v^{\frac{1}{2}}\,dv-1600\int_{1600}^{u+1600}v^{-\frac{1}{2}}\,dv\right)\\&=80\left(\left[\frac{2}{3}v^{\frac{3}{2}}\right]_{1600}^{u+1600}-1600\left[2\sqrt{v}\right]_{1600}^{1600+u}\right)\\&=80\left(\frac{2}{3}\left(s^3-40^3\right)-3200\left(s-40\right)\right)\\&=\frac{160}{3}(s-40)\left(\left(s^2+40s+40^2\right)-4800\right)\end{aligned}$$

The variance of the loss is therefore

$$\begin{aligned}\frac{160}{3}(s-40)\left(\left(s^2+40s+40^2\right)-4800\right)-80^2(s-40)^2 &= \frac{160}{3}(s-40)\left(\left(s^2+40s+40^2\right)-4800-120(s-\right.\\ &= \frac{160}{3}(s-40)\left(s^2-80s+40^2\right)\\ &= \frac{160}{3}(s-40)^3\end{aligned}$$

The aggregate loss therefore has mean $32\times 80\,(s-40) = 2560(s-40)$ and variance $32\times\frac{160}{3}\,(s-40)^3 + 5152\times(80(s-40))^2$. For the Pareto approximation to aggregate losses, the parameters are given by solving

$$\begin{aligned}\frac{\theta}{\alpha-1} &= 2560(s-40)\\ \frac{\theta^2\alpha}{(\alpha-1)^2(\alpha-2)} &= \frac{5120}{3}(s-40)^3+32972800(s-40)^2\\ \frac{\alpha}{\alpha-2} &= \frac{5120}{3}(s-40)^3+32972800(s-40)^2(2560(s-40))^2\\ &= \frac{(s-40)+19320}{3840}\\ &= \frac{s+19280}{3840}\\ \alpha &= \frac{2}{1-\frac{3840}{s+19280}} = \frac{2s+38560}{s+15440}\\ \theta &= 2560\frac{s+23120}{s+15440}(s-40)\end{aligned}$$

Setting $w=\frac{s}{80}$, this becomes

$$\begin{aligned}\alpha &= \frac{2w+482}{w+193}\\ \theta &= 102400\frac{w+289}{w+193}(2w-1)\end{aligned}$$

Under this model, the expected payment on the reinsurance policy is

$$\begin{aligned}\int_a^\infty\left(\frac{\theta}{\theta+x}\right)^\alpha dx &= \int_{\theta+a}^\infty \theta^\alpha u^{-\alpha}\,du\\ &= \frac{\theta^\alpha}{\alpha-1}\left[-u^{1-\alpha}\right]_{\theta+a}^\infty\\ &= \frac{\theta^\alpha}{\alpha-1}\left(\theta+a\right)^{1-\alpha}\end{aligned}$$

We have $a = \frac{\theta}{\alpha-1}$, so the expected payment on the reinsurance is

$$\begin{aligned}
\frac{\theta^\alpha}{\alpha-1}(\theta+a)^{1-\alpha} &= \frac{\theta^\alpha}{\alpha-1}\left(\theta\left(1+\frac{1}{\alpha-1}\right)\right)^{1-\alpha} \\
&= \frac{\theta}{\alpha-1}\left(\frac{\alpha}{\alpha-1}\right)^{1-\alpha} \\
&= \frac{\theta}{\alpha-1}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}
\end{aligned}$$

Substituting the equations

$$\begin{aligned}
\alpha &= \frac{2w+482}{w+193} \\
\alpha - 1 &= \frac{w+289}{w+193} \\
\theta &= 102400\frac{w+289}{w+193}(2w-1) \\
\frac{\theta}{\alpha-1} &= 102400(2w-1) \\
\frac{\alpha-1}{\alpha} &= \frac{w+289}{2(w+241)} \\
\frac{\theta}{\alpha-1}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} &= 102400(2w-1)\left(\frac{w+289}{2(w+241)}\right)^{\frac{w+289}{w+193}}
\end{aligned}$$

Numerically solving this

$$\frac{\theta}{\alpha-1}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} = 102400(2w-1)\left(\frac{w+289}{2(w+241)}\right)^{\frac{w+289}{w+193}} = 500000$$

gives $w = 5.74312$, which gives $s = 80 \times 5.74312 = 459.4496$ and $u = s^2 - 1600 = 459.4496^2 - 1600 = \$209,493.93$.