ACSC/STAT 4703, Actuarial Models II

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Homework Sheet 1

Model Solutions

Basic Questions

1. *An insurance company models losses as following a gamma distribution with $\alpha = 0.8$ and $\theta = 2000$. The fixed expenses are \$300 per claim, and variable expenses are 12% of loss amount. What is the density function of the distribution of the total cost to the insurance company for a random loss?*

   If the loss amount is X, then the insurer pays a total of $1.12X + 300$. $1.12X$ follows a gamma distribution with $\alpha = 0.8$ and $\theta = 2240$. The density function is therefore

$$f(x) = \begin{cases} \frac{(x-300)^{-0.2}e^{-\frac{x}{2240}}}{2240^{0.8}\Gamma(0.8)} & \text{if } x > 300 \\ 0 & \text{otherwise} \end{cases}$$

2. *An insurer models the area affected by an earthquake as a circle with radius (in km) following a Pareto distribution with $\alpha = 1.4$ and $\theta = 5$. What is the density function for the distribution of the affected area in km^2?*

   The area of a circle is πr^2, so the we need the density function

$$\begin{aligned} f_{\pi R^2}(x) &= \frac{f_R\left(\sqrt{\frac{x}{\pi}}\right)}{2\sqrt{\pi x}} \\ &= \frac{\frac{1.4\times 5^{1.4}}{\left(5+\sqrt{\frac{x}{\pi}}\right)^{2.4}}}{2\sqrt{\pi x}} \\ &= \frac{0.7\times 5^{1.4}}{\sqrt{\pi x}\left(5+\sqrt{\frac{x}{\pi}}\right)^{2.4}} \\ &= \frac{3.75907602693}{x^{0.5}\left(5+\sqrt{\frac{x}{\pi}}\right)^{2.4}} \end{aligned}$$

3. *An insurance company has the following data on its policies:*

Policy limit	*Losses Limited to*				
	50,000	*100,000*	*200,000*	*500,000*	*1,000,000*
50,000	*3,194,726*				
100,000	*5,586,215*	*10,503,540*			
200,000	*8,947,072*	*30,793,171*	*37,895,098*		
500,000	*5,354,052*	*12,769,853*	*16,108,054*	*18,450,094*	
1,000,000	*2,854,741*	*11,529,017*	*15,416,701*	*19,129,888*	*20,171,889*

Use this data to calculate the ILF from $50,000 to $1,000,000 using

(a) The direct ILF estimate.

The direct ILF estimate is $\frac{20171889}{2854741} = 7.06610126803$

(b) The incremental method.

The incremental ILF is

$$\frac{20171889}{19129888} \times \frac{18450094 + 19129888}{16108054 + 15416701} \times \frac{37895098 + 16108054 + 15416701}{30793171 + 12769853 + 11529017} \times \frac{10503540 + 30793171 + 12769853 + 11529017}{5586215 + 8947072 + 5354052 + 2854741} = 4.56854826961$$

4. *An insurance company charges a risk charge equal to the square of the average loss amount, divided by 50,000. It has the following loss data on a set of 7,420 policies with limit $2,000,000.*

Losses Limited to	*500,000*	*1,000,000*	*2,000,000*
Total claimed	*$9,950,249*	*$13,383,022*	*$15,040,978*

Calculate the ILF from $500,000 to $1,000,000.

The pure premium for limit $500,000 is $\frac{9950249}{7420} = 1341.00390836$. The risk charge is $\frac{1341.00390836^2}{50000} = 35.9658296448$. Thus the total premium is $1341.00390836 + 35.9658296448 = \$1376.969738$. The pure premium for limit $1,000,000 is $\frac{13383022}{7420} = 1803.64177898$. The risk charge is $\frac{1803.64177898^2}{50000} = 65.0624733376$. Thus the total premium is $1803.64177898 + 65.0624733376 = 1868.70425232$. The ILF is therefore $\frac{1868.70425232}{1376.969738} = 1.35711352309$.

Standard Questions

5. *A health insurance company models claims as being either* preventative *or* curative. *84% of claims are preventative. Costs for preventative claims are modelled as following a gamma distribution with* $\alpha = 2.6$ *and* $\theta = 180$. *Costs for curative claims are broken into* diagnostic costs *which are modelled following a Pareto distribution with* $\alpha = 1$ *and* $\theta = 170$ *and* treatment costs, *which are modelled as independent of diagnostic costs,*

and following a Pareto distribution with $\alpha = 1$ and $\theta = 800$. What is the density function for the total cost of a random claim?

For curative claims, the total cost is the sum of the diagnostic costs and the treatment costs, namely a Pareto distribution with $\alpha = 1$ and $\theta = 170$ and a Pareto distribution with $\alpha = 1$ and $\theta = 800$. The density of this sum is given by the convolution

$$\begin{aligned}
f_C(x) &= \int_0^x f_D(y) f_P(x-y)\,dy \\
&= \int_0^x \frac{170}{(170+y)^2}\frac{800}{(800+x-y)^2}\,dy \\
&= 136000\int_0^x \left(\frac{1}{(170+y)\,(800+x-y)}\right)^2\,dy \\
&= \frac{136000}{(970+x)^2}\int_0^x \left(\frac{1}{(170+y)}+\frac{1}{(800+x-y)}\right)^2\,dy \\
&= \frac{136000}{(970+x)^2}\int_0^x \left(\frac{1}{(170+y)^2}+\frac{1}{(800+x-y)^2}+\frac{2}{970+x}\left(\frac{1}{(170+y)}+\frac{1}{(800+x-y)}\right)\right)^2\,dy \\
&= \frac{136000}{(970+x)^2}\left(\int_{170}^{x+170} u^{-2}+\frac{2u^{-1}}{970+x}\,du+\int_{800}^{800+x} v^{-2}+\frac{2v^{-1}}{970+x}\,dv\right) \\
&= \frac{136000}{(970+x)^2}\left(\frac{1}{170}-\frac{1}{170+x}+\frac{2\log\left(\frac{170+x}{170}\right)}{970+x}+\frac{1}{800}-\frac{1}{800+x}+\frac{2\log\left(\frac{800+x}{800}\right)}{970+x}\right) \\
&= \frac{136000}{(970+x)^2}\left(\frac{970}{136000}-\frac{1}{170+x}-\frac{1}{800+x}\right)+\frac{272000}{(970+x)^3}\log\left(\frac{(170+x)(800+x)}{136000}\right)
\end{aligned}$$

The density of the mixture distribution is therefore

$$\begin{aligned}
&0.84 f_R(x) + 0.16 f_C(x) \\
&= 0.84\frac{x^{1.6}e^{-\frac{x}{180}}}{180^{2.6}\Gamma(2.6)} + 0.16\left(\frac{136000}{(970+x)^2}\left(\frac{970}{136000}-\frac{1}{170+x}-\frac{1}{800+x}\right)+\frac{272000}{(970+x)^3}\log\left(\frac{(170+x)(800+x)}{136000}\right)\right)
\end{aligned}$$

6. *The pure premium ILF from $1,000,000 to $2,000,000 is 1.2. A reinsurer offers excess-of-loss reinsurance of $1,000,000 over $1,000,000 for a loading of 25%. An insurer whose premium includes a 10% loading on expected claims and a risk charge equal to the square of the expected claims divided by $100,000 can reduce its premium for policies with limit $2,000,000 by 5% by buying reinsurance. What was the premium for policies with limit $2,000,000 before buying the reinsurance? [It is not 0.]*

Let C be the expected claim for a policy with limit \$1,000,000. The expected claim for a policy with limit \$2,000,000 is therefore $1.2C$. The reinsurer's premium for the excess-of-loss insurance is therefore, $0.2C \times 1.25 = 0.25C$. The insurer's premium for policies with limit \$1,000,000 is $1.1C + \frac{C^2}{100000}$, and the premium for policies with limit \$2,000,000 is $1.32C + \frac{1.44C^2}{100000}$ without reinsurance, and $1.1C + \frac{C^2}{100000} + 0.25C$ with reinsurance. Since the reinsurance allows a 5% reduction in premium, we have

$$\begin{aligned}
1.1C + \frac{C^2}{100000} + 0.25C &= 0.95\left(1.32C + \frac{1.44C^2}{100000}\right)\\
1.35C + \frac{C^2}{100000} &= 1.254C + \frac{1.368C^2}{100000}\\
0.096C &= \frac{0.368C^2}{100000}\\
C &= \frac{0.096 \times 100000}{0.368}\\
&= 26086.9565217
\end{aligned}$$

Thus the premium for policies with limit \$2,000,000 without reinsurance is

$$26086.9565217 \times 1.32 + \frac{1.44 \times 26086.9565217^2}{100000} = \$44,234.40$$