ACSC/STAT 4703, Actuarial Models II

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Homework Sheet 2

Model Solutions

Basic Questions

1. *An insurer models losses as following a distribution with distribution function $F(x) = \frac{x^3+x^2+x}{x^3+x^2+5x+1}$. They find that $c_n = n^{\frac{1}{2}}$ and $d_n = n^{\frac{1}{2}}$ make the distribution of block maxima converge. What is the limiting distribution?*

   We have $P(M_n < c_nx+d_n) = F(c_nx+d_n)^n = \left(\frac{(c_nx+d_n)^3+(c_nx+d_n)^2+(c_nx+d_n)}{(c_nx+d_n)^3+(c_nx+d_n)^2+5(c_nx+d_n)+1}\right)^n$. Substituting the given values gives

$$\begin{aligned}\log\left(P(M_n < c_nx+d_n)\right) &= n\log\left(\frac{(c_nx+d_n)^3+(c_nx+d_n)^2+(c_nx+d_n)}{(c_nx+d_n)^3+(c_nx+d_n)^2+5(c_nx+d_n)+1}\right)\\ &= n\log\left(\frac{n^{\frac{3}{2}}(x+1)^3+n(x+1)^2+n^{\frac{1}{2}}(x+1)}{n^{\frac{3}{2}}(x+1)^3+n(x+1)^2+5n^{\frac{1}{2}}(x+1)+1}\right)\\ &= n\log\left(1-\frac{4n^{\frac{1}{2}}(x+1)+1}{n^{\frac{3}{2}}(x+1)^3+n(x+1)^2+5n^{\frac{1}{2}}(x+1)+1}\right)\\ &\to n\left(-\frac{4n^{\frac{1}{2}}(x+1)+1}{n^{\frac{3}{2}}(x+1)^3+n(x+1)^2+5n^{\frac{1}{2}}(x+1)+1}\right)\\ &= -\frac{4(x+1)+n^{-\frac{1}{2}}}{(x+1)^3+n^{-\frac{1}{2}}(x+1)^2+5n^{-1}(x+1)+n^{-\frac{3}{2}}}\\ &= -4(1+x)^{-2}+O\left(n^{-\frac{1}{2}}\right)\end{aligned}$$

   Thus, the limiting distribution is Fréchet, with $\xi = 2$.

2. *An insurer models losses as following a distribution with survival function $S(x) = 1-e^{-\frac{1}{x}-\frac{1}{x^2}}$. What values of c_n and d_n make the distribution of block maxima converge, and what is the limiting distribution?*

   We have $nS(c_nx+d_n) = n\left(1-e^{-\frac{1}{x}-\frac{1}{x^2}}\right)$. We want this to converge for every x. For $x=0$, we want $n\left(1-e^{-\frac{1}{d_n}-\frac{1}{d_n^2}}\right)$ to converge. This is easily seen to be achieved by $d_n = n$. Similarly, we see that $c_n = n$ gives

$$\lim_{n\to\infty} nS(c_nx+d_n) = \lim_{n\to\infty} n\left(1-e^{-\frac{1}{n(x+1)}-\frac{1}{n^2(x+1)^2}}\right)$$
$$= \lim_{n\to\infty} n\left(1-\left(1-\left(\frac{1}{n(x+1)}+\frac{1}{n^2(x+1)^2}\right)+\frac{1}{2}\left(\frac{1}{n(x+1)}+\frac{1}{n^2(x+1)^2}\right)^2-\ldots\right)\right)$$
$$= \frac{1}{x+1}$$

3. *A loss follows a distribution from the MDA of a Gumbel distribution. A reinsurer estimates that the probability of the loss exceeding \$1,000,000 is 0.005. The expected payment on an excess-of-loss reinsurance contract of \$1,000,000 over \$1,000,000 for this loss is \$911.40. What is the expected payment on an excess-of-loss reinsurance contract of \$2,000,000 over \$1,000,000.*

Since the distribution of X is in the MDA of a Gumbel distribution, the excess-loss function converges to an exponential distribution. We also have $\mathbb{E}((X-1000000)\wedge 1000000|X>1000000) = \frac{911.40}{0.005} = \$182280$, which gives the scale parameter θ of the excess-loss distribution by solving

$$\int_0^{1000000} e^{-\frac{x}{\theta}}\,dx = 182280$$
$$\theta\left(1-e^{-\frac{1000000}{\theta}}\right)\,dx = 182280$$

For an excess-of-loss reinsurance of \$2,000,000 over \$1,000,000, the expected payment conditional on a payment being made is

$$\int_0^{2000000} e^{-\frac{x}{\theta}}\,dx = \theta\left(1-e^{-\frac{2000000}{\theta}}\right)$$
$$= \theta\left(1-e^{-\frac{1000000}{\theta}}\right)\left(1+e^{-\frac{1000000}{\theta}}\right)$$
$$= 42280\left(1+e^{-\frac{1000000}{\theta}}\right)$$

Numerically, we get $\theta = 183056$, so the overall expected payment is $0.005\times 182280\left(1+e^{-\frac{1000000}{183056}}\right) = \$915.27$

Standard Questions

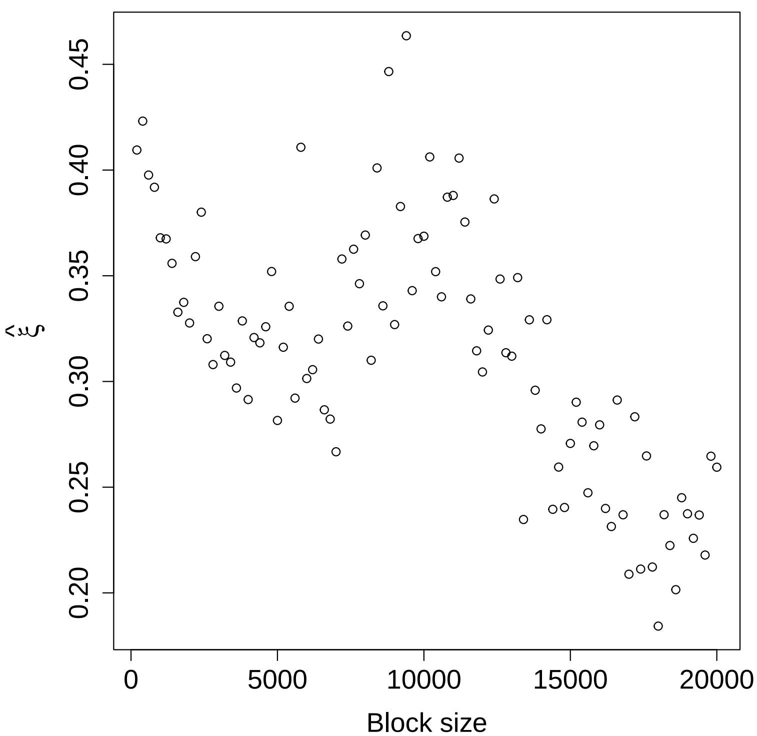
4. *The file* `HW2_data.txt` *contains 1,000,000 values of a random variable.*

   (a) *By dividing into blocks of different sizes, and using the* `fit.GEV` *function in the* `QRM` *package in* `R`*, estimate the tail index ξ.*

We use the following `R` code to evaluate for block sizes multiples of 200 up to 20,000 (If the block size is too large, there are too few observations and `fit.GEV` produces an error):

```
HW2Q4<-read.table("HW2_data.txt")[[1]]
library("QRM")
GEV_estimates<-rep(0,100)
for(i in seq_len(100)){
    nbl<-floor(5000/i)
    M<-matrix(HW2Q4[seq_len(nbl*200*i)],200*i,nbl)
    GEV_model<-fit.GEV(apply(M,2,max))
    GEV_estimates[i]<-GEV_model$par.ests["xi"]
}
```

This produces the following estimates:

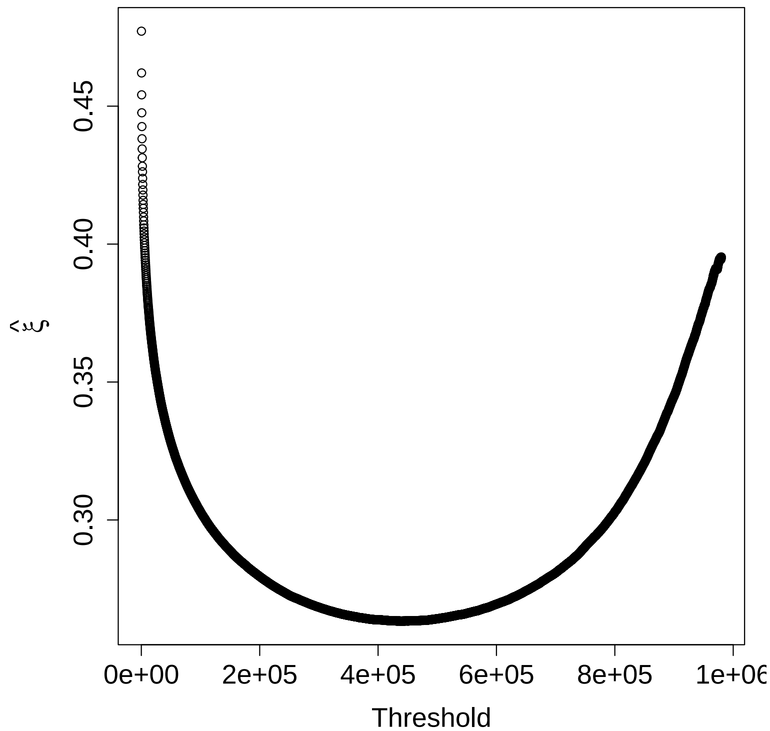


(b) *Use the Hill estimator to estimate ξ at a range of different thresholds, from the data in the file* `HW2_data.txt`.

We use the following `R` code to evaluate for threshold positions multiples of 200 up to 980,000:

```
HW2Q4.log.sort<-sort(log(HW2Q4))
Hill_estimates<-rep(0,4900)
for(i in seq_len(4900)){
    pos<-i*200
    Hill_estimates[i]<-mean(HW2Q4.log.sort[(pos+1):1000000])-HW2Q4.log.sort[pos]
}
```

This produces the following estimates:



We see that the estimates are stable for thresholds that are not too small or too large.

5. *A insurer wants to calculate the ILF for a heavy-tailed loss. Based on previous data, they estimate that the distribution of the loss is in the MDA of a Weibull EV distribution with $\xi = -1$. The ILF from \$1,000,000 to \$2,000,000 is 1.18 and the ILF from \$2,000,000 to \$5,000,000 is 1.39. Assuming the GPD approximation applies to losses above \$1,000,000, what is the ILF from \$5,000,000 to \$10,000,000?*

Under the GPD approximation, losses exceeding \$1,000,000 follow a GPD distribution with parameter ξ. The survival function is therefore $\left(1+\xi\frac{x}{\beta}\right)^{-\frac{1}{\xi}}$.

We cannot use this approximation to estimate $\mathbb{E}(X \wedge 1000000)$, but we have that

$$\begin{aligned}
\mathbb{E}(((X \wedge b) - a)|X > a) &= \int_0^{b-a} S_{x-a}(x)\,dx \\
&= \int_0^{b-a} \left(1 - \frac{x}{\beta}\right)\,dx \\
&= \beta\left[\frac{1}{2}u^2\right]_{1-\frac{(b-a)}{\beta}}^1 \\
&= \frac{\beta}{2}\left(1 - \left(1 - \frac{(b-a)}{\beta}\right)^2\right) \\
&= \frac{\beta}{2}\left(1 - \left(1 - 2\frac{(b-a)}{\beta} + \left(\frac{(b-a)}{\beta}\right)^2\right)\right) \\
&= \beta\left(\frac{(b-a)}{\beta} - \frac{1}{2}\left(\frac{(b-a)}{\beta}\right)^2\right) \\
&= b - a - \frac{\beta}{2}(b-a)^2
\end{aligned}$$

Let l_0 be the expected loss with policy limit \$1,000,000 and s_0 be the probability of a loss exceeding \$1,000,000. Since β is a scale parameter, we can rescale the loss in units of \$1,000,000. We have

$$\begin{aligned}
\mathbb{E}(((X \wedge 2) - 1)_+) &= s_0\mathbb{E}(((X \wedge 2) - 1)|X > 1) \\
&= s_0\left(1 - \frac{\beta}{2}\right) = 0.18l_0 \\
\mathbb{E}(((X \wedge 5) - 1)_+) &= s_0\,(4 - 8\beta) = (1.18 \times 1.39 - 1)l_0 = 0.6402l_0 \\
4 - 8\beta &= \frac{0.6402}{0.18}\left(1 - \frac{\beta}{2}\right) \\
1.1199\beta &= 0.0798 \\
\beta &= 0.0712563621752
\end{aligned}$$

Using this, we calculate

$$s_0\left(1-\frac{\beta}{2}\right)=0.18l_0$$

$$\frac{l_0}{s_0}=\frac{1-\frac{0.0712563621752}{2}}{0.18}=5.35762121618$$

$$\mathbb{E}(((X\wedge 10)-1)_+)=s_0\left(9-\frac{0.0712563621752\times 81}{2}\right)=6.1141173319s_0=1.1412l_0$$

So the ILF from \$1,000,000 to \$10,000,000 is 2.1412. The ILF from \$1,000,000 to \$5,000,000 is $1.18\times 1.39=1.6402$, so the ILF from \$5,000,000 to \$10,000,000 is $\frac{2.1412}{1.6402}=1.30545055481$