ACSC/STAT 4703, Actuarial Models II

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Homework Sheet 3

Model Solutions

Basic Questions

1. *Loss amounts follow a gamma distribution with shape $\alpha = 3.4$ and scale $\theta = 700$. The distribution of the number of losses is given in the following table:*

Number of Losses	*Probability*
0	*0.750*
1	*0.114*
2	*0.085*
3	*0.051*

Assume all losses are independent and independent of the number of losses. The insurance company buys excess-of-loss reinsurance on the part of the loss above \$7,000. Calculate the expected payment for this excess-of-loss reinsurance.

If there are n claims, then the total losses follow a gamma distribution with shape $\alpha = 3.4n$ and $\theta = 700$. The expected payment on the excess of loss distribution in this case is therefore given by:

$$\begin{aligned}\mathbb{E}((X-7000)_+) &= \frac{700}{\Gamma(4.4n)}\int_{\frac{7000}{700}}^{\infty}\left(x-\frac{7000}{700}\right)x^{3.4n-1}e^{-x}\,dx\\ &= \frac{700}{\Gamma(3.4n)}\left(\int_{10}^{\infty}x^{3.4n}e^{-x}\,dx - 10\int_{10}^{\infty}x^{3.4n-1}e^{-x}\,dx\right)\\ &= 700\,(3.4n\texttt{pgamma(10,shape=3.4n+1,lower.tail=FALSE)}\\ &\qquad -10\texttt{pgamma(10,shape=3.4n,lower.tail=FALSE)})\end{aligned}$$

This gives the following table

n	$P(N=n)$	$\mathbb{E}((S-7000)_+\|N=n)$	$\mathbb{E}((S-7000)_+I_{N=n})$
0	0.750	0.00000	0.000000
1	0.114	4.246378	0.4840871
2	0.085	145.415855	12.3603476
3	0.051	950.542653	48.4776753

So the total expected payment is $0.4840871 + 12.3603476 + 48.4776753 =$ \$61.32.

2. *Loss frequency follows a negative binomial distribution with $r = 7$ and $\beta = 0.25$. Loss severity (in thousands) has the following distribution:*

Severity	*Probability*
0	*0.44*
1	*0.27*
2	*0.11*
3	*0.09*
4 or more	*0.09*

Use the recursive method to calculate the exact probability that aggregate claims are at least \$4,000.

Recall that for the negaive binomial distribution, $a = \frac{\beta}{1+\beta} = \frac{0.25}{1.25} = 0.2$ and $b = \frac{(r-1)\beta}{1+\beta} = 1.2$. We compute $f_S(0) = P_S(0) = P_N(P_X(0)) = (1 + 0.25 - 0.25f_X(0))^{-7} = (1 + 0.25 - 0.25 \times 0.44)^{-7} = 0.39963732253$

The recurrence formula is

$$f(x) = \frac{\sum_{k=1}^{x} \left(0.2 + 1.2\frac{k}{x}\right) f_X(k)f(x-k)}{1 - 0.2 \times 0.44} = 0.219298245614 \sum_{k=1}^{x} \left(1 + 6\frac{k}{x}\right) f_X(k)f(x-k)$$

Applying this gives:

$$\begin{aligned}
f(1) &= 0.219298245614 \times 7 \times 0.27 \times 0.39963732253 = 0.165639153417 \\
f(2) &= 0.219298245614\,(4 \times 0.27 \times 0.165639153417 + 7 \times 0.11 \times 0.39963732253) = 0.106712943868 \\
f(3) &= 0.219298245614\,(3 \times 0.27 \times 0.106712943868 + 5 \times 0.11 \times 0.165639153417 + 7 \times 0.09 \times 0.39963732253) \\
&= 0.0941470465147
\end{aligned}$$

The probability that aggregate claims are at least \$4,000 is therefore

$$\begin{aligned}
&1 - f(0) - f(1) - f(2) - f(3) \\
=&1 - 0.39963732253 - 0.165639153417 - 0.106712943868 - 0.0941470465147 \\
=&0.23386353367
\end{aligned}$$

3. *Use an arithmetic distribution ($h = 1$) to approximate a Burr distribution with $\alpha = 3$, $\gamma = 2$ and $\theta = 1$.*

   (a) Using the method of rounding, calculate the mean of the arithmetic approximation. [You can evaluate this numerically: use 5,000 terms in the sum.]

Using the method of rounding, we set

$$\begin{aligned} p_0 &= P\left(X < \frac{1}{2}\right) \\ &= 1 - \frac{\theta^{\alpha\gamma}}{\left(\theta^\gamma + \left(\frac{1}{2}\right)^\gamma\right)^\alpha} \\ &= 1 - \frac{1}{\left(1 + 0.5^2\right)^3} \\ &= 0.488 \end{aligned}$$

and

$$\begin{aligned} p_n &= P\left(n - \frac{1}{2} \leqslant X < n + \frac{1}{2}\right) \\ &= \frac{1}{\left(\left(n - \frac{1}{2}\right)^2 + 1^2\right)^3} - \frac{1}{\left(\left(n + \frac{1}{2}\right)^2 + 1^2\right)^3} \\ &= \frac{1}{\left(n^2 - n + 1.25\right)^3} - \frac{1}{\left(n^2 + n + 1.25\right)^3} \\ &= \frac{\left(n^2 + n + 1.25\right)^3 - \left(n^2 - n + 1.25\right)^3}{\left(n^2 - n + 1.25\right)^3 \left(n^2 + n + 1.25\right)^3} \\ &= \frac{2n\left(\left(n^2 + n + 1.25\right)^2 + \left(n^2 + n + 1.25\right)\left(n^2 - n + 1.25\right) + \left(n^2 - n + 1.25\right)^2\right)}{\left(n^2 - n + 1.25\right)^3 \left(n^2 + n + 1.25\right)^3} \\ &= \frac{2n\left(3n^4 + 9.5n^2 + 4.6875\right)}{\left(\left(n^2 + 1.25\right)^2 - n^2\right)^3} \\ &= \frac{2n\left(3n^4 + 9.5n^2 + 4.6875\right)}{\left(n^4 + 1.5n^2 + 1.5625\right)^3} \end{aligned}$$

We can also calculate so $S_a(n) = S_x\left(n - \frac{1}{2}\right) = \frac{1}{((n-0.5)^2+1^2)^3}$ and

$$\mathbb{E}(X_a) = \sum_{n=1}^{\infty} S_X\left(n - \frac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{1}{\left(\left(n - \frac{1}{2}\right)^2 + 1^2\right)^3} = \sum_{n=1}^{\infty} \frac{1}{\left(n^2 - n + 1.25\right)^3}$$

We numerically evaluate this as 0.5443451 (summing the first 5000 terms).

(b) Using the method of local moment matching, matching 1 moment on each interval, estimate the probability that the value is larger than 4.5.

We have

$$1-(p_0+p_1+p_2+p_3+p_{4,l}) = S_X(4) = \frac{1}{(4^2+1)^3} = 0.000203541624262$$

and

$$\begin{aligned}
p_{4,u}+p_{5,l} &= \frac{1}{(4^2+1)^3} - \frac{1}{(5^2+1)^3} \\
&= 0.000146645857307 \\
4p_{4,u}+5p_{5,l} &= \int_4^5 \frac{6x^2}{(x^2+1)^4}\,dx \\
&= 6\int_4^5 \frac{x^2+1-1}{(x^2+1)^4}\,dx \\
&= 6\int_4^5 (x^2+1)^{-3} - (x^2+1)^{-4}\,dx \\
&= 6\int_{\tan^{-1}(4)}^{\tan^{-1}(5)} \left((\tan^2(\theta)+1)^{-3} - (\tan^2(\theta)+1)^{-4}\right)\sec^2(\theta)\,d\theta \\
&= 6\int_{\tan^{-1}(4)}^{\tan^{-1}(5)} \left(\cos^6(\theta) - \cos^8(\theta)\right)\sec^2(\theta)\,d\theta \\
&= 6\int_{\tan^{-1}(4)}^{\tan^{-1}(5)} \cos^4(\theta) - \cos^6(\theta)\,d\theta \\
&= \frac{6}{32}\int_{\tan^{-1}(4)}^{\tan^{-1}(5)} \cos(2\theta) - 2\cos(4\theta) - \cos(6\theta)\,d\theta \\
&= \frac{3}{32}[\sin(2\theta)]_{\tan^{-1}(4)}^{\tan^{-1}(5)} - \frac{3}{32}[\sin(4\theta)]_{\tan^{-1}(4)}^{\tan^{-1}(5)} - \frac{1}{32}[\sin(6\theta)]_{\tan^{-1}(4)}^{\tan^{-1}(5)} + \frac{6}{32}[\theta]_{\tan^{-1}(4)}^{\tan^{-1}(5)} \\
&= 0.0006423467
\end{aligned}$$

where we have used the following:

$$\begin{aligned}
\cos(6\theta) &= \cos^6(\theta) - 15\cos^4(\theta)\sin^2(\theta) + 15\cos^2(\theta)\sin^4(\theta) - \sin^6(\theta) \\
&= \cos^6(\theta) - 15\cos^4(\theta)(1-\cos^2(\theta)) + 15\cos^2(\theta)(1-\cos^2(\theta))^2 - (1-\cos^2(\theta))^3 \\
&= 32\cos^6(\theta) - 48\cos^4(\theta) + 18\cos^2(\theta) - 1 \\
\cos(4\theta) &= \cos^4(\theta) - 6\cos^2(\theta)\sin^2(\theta) + \sin^4(\theta) \\
&= \cos^4(\theta) - 6\cos^2(\theta)(1-\cos^2(\theta)) + (1-\cos^2(\theta))^2 \\
&= 8\cos^4(\theta) - 8\cos^2(\theta) + 1 \\
\cos(6\theta) + 2\cos(4\theta) &= 32\cos^6(\theta) - 32\cos^4(\theta) + 2\cos^2(\theta) + 1 \\
\cos(6\theta) + 2\cos(4\theta) - \cos(2\theta) + 2 &= 32(\cos^6(\theta) - \cos^4(\theta))
\end{aligned}$$

[Alternatively, we can use partial fractions to calculate the integral, or evaluate it numerically.]

So

$$p_{4,u} = 5 \times 0.000146645857307 - 0.0006423467 = 0.000090882586535$$

Thus, $P(X_a > 4.5) = 0.000203541624262 - 0.000090882586535 = 0.000112659037727$.

Standard Questions

4. *The number of claims an insurance company receives follows a compound Poisson-negative binomial distribution with $\lambda = 2099$ for the primary distribution and $r = 0.7, \beta = 1.3$ for the secondary distribution. Claim severity follows a Poisson distribution with $\lambda = 5$. Calculate the probability that aggregate losses exceed $10,000.*

   (a) Starting the recurrence 6 standard deviations below the mean [You need to calculate 15,000 terms of the recurrence for f_S.]

   We compute the intermediate distribution A by the usual recurrence, noting that $f_A(0) = P_A(0) = (2.3 - 1.3 f_X(0))^{-0.7} = (2.3 - 1.3 \times e^{-5})^{-0.7} = 0.559693809593$ and the recurrence is

$$f_A(x) = \frac{\frac{13}{23}}{1 - \frac{13}{23}e^{-5}} \sum_{k=1}^{x} \left(1 - 0.3\frac{k}{x}\right) f_X(k) f_A(n-k) = 0.567378197168 \sum_{k=1}^{x} \left(1 - 0.3\frac{k}{x}\right) f_X(k) f_A(n-k)$$

We therefore compute the distribution of A using the usual recurrence:

```
fx<-dpois(seq_len(50000),lambda=5)
### Since f_X(0) is not used in the recurrence, I have started the
### vector fx at f_X(1). This makes the indices slightly easier.

fA<-rep((2.3-1.3*exp(-5))^(-0.7),50000)
for(i in seq_len(49999)){
    fA[i+1]<-13/(23-13*exp(-5))*sum((i-0.3*seq_len(i))*fx[seq_len(i)]*fA[i:1])/i
    ##Note that fA[i+1]=f_A(i), as this vector includes 0.
}
sum(fA) # check we have enough terms.
```

The mean and variance of A are given by the standard formulae:

$$\mathbb{E}(A) = 0.7 \times 1.3 \times 5 = 4.55$$
$$\text{Var}(A) = 0.7 \times 1.3 \times 5 + 0.7 \times 1.3 \times 2.3 \times 5^2 = 56.875$$

For the distribution of S, the recurrence is

$$f_S(x) = \sum_{k=1}^{x} 2099\frac{k}{x} f_A(k) f_S(n-k)$$

$f_S(0)$ is too small to start at zero. Therefore, we start the recurrence 6 standard deviations below the mean. The mean and standard deviation are given by

$$\mathbb{E}(S) = 2099 \times 4.55 = 9550.45$$
$$\mathrm{Var}(A) = 2099 \times 56.875 + 2099 \times 4.55^2 = 162835.1725$$

so 6 standard deviations below the mean is $9550.45 - 6\sqrt{162835.1725} = 7129.27957849$, so we start the recurrence from $f_S(7129) = 0$ and $f_S(7130) = 1$

```
fS<-rep(0,50000)
fS[7130]<-1 # Since we are truncating 0, we can let fS[1]=f_S(1)

for(i in seq_len(30000)){
    fS[7130+i]<-2099/(7130+i)*sum(seq_len(i)*fA[seq_len(i)+1]*fS[7129+(i:1)])
}

fS[37100] # check we have enough terms - this should be negligible.
fS<-fS/sum(fS) # rescale.
sum(fS[10001:37100]) #answer to question.
```

This gives the probability that $S > 10000$ as 0.1328735.

(b) Using a suitable convolution.

We can use the same code to get the distribution of f_A. Now we express S as a sum $S_1 + \cdots + S_8$, where S_i has a compound distribution with secondary distribution A, and primary distribution Poisson with mean 262.375. We compute the distribution of S_i using the standard recurrence

$$f_{S_i}(x) = \sum_{k=1}^{x} 262.375\frac{k}{x} f_A(k) f_{S_i}(n-k)$$

with $f_{S_i}(0) = e^{262.375(f_A(0)-1)} = e^{262.375\times(0.559693809593-1)} = 6.72951472419\times 10^{-51}$

```r
fSi<-rep(exp(262.375*(fA[1]-1)),10001)
for(i in seq_len(10000)){ #10000 should be enough points
    fSi[i+1]<-262.375/i*sum(seq_len(i)*fA[seq_len(i)+1]*fSi[i:1])
}

sum(fSi) # check that we have enough points

ConvolveSelf<-function(n){
    convolution<-vector("numeric",2*length(n))
    for(i in 1:(length(n))){
        convolution[i]<-sum(n[1:i]*n[i:1])
    }
    for(i in 1:(length(n))){
      convolution[2*length(n)+1-i]<-sum(n[length(n)+1-(1:i)]*n[length(n)+1-(i:1)])
    }
    return(convolution)
}

### Convolve 8 times
fSi2<-ConvolveSelf(fSi)
fSi4<-ConvolveSelf(fSi2)
fSi8<-ConvolveSelf(fSi4)

sum(fSi8)
sum(fSi8[10002:80000]) # gives the same answer as (a)
### Compare results in (a) and (b)
max(abs(fSi8[seq_len(35000)+1]-fS[seq_len(35000)]))
```

This gives the probability that $S > 10000$ as 0.1328735.

[The maximum difference in estimated probabilies between these two methods is 1.760321×10^{-11}.]