ACSC/STAT 4703, Actuarial Models II

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Homework Sheet 5

Model Solutions

1. *A home insurance company classifies policyholders as high-risk, medium-risk or low-risk . Annual claims from high-risk policyholders follow a Pareto distribution with $\alpha = 2.4$ and $\theta = 830$. Annual claims from medium-risk policyholders follow a Pareto distribution with $\alpha = 4.6$ and $\theta = 960$. Annual claims from low-risk policyholders follow a gamma distribution with $\alpha = 4$ and $\theta = 59$. 16% of policyholders are high risk, 68% are medium risk and 16% are low risk.*

   (a) Calculate the expectation and variance of the aggregate annual claims from a randomly chosen policyholder.

   - For a high-risk policyholder, the expected claim is $\frac{830}{1.4} = 592.857142857$. The variance is $\frac{2.4\times 830^2}{1.4^2\times 0.4} = 2108877.55102$
   - For a medium-risk policyholder, the expected claim is $\frac{960}{3.6} = 266.666666667$. The variance is $\frac{960^2\times 4.6}{3.6^2\times 2.6} = 125811.965812$
   - For a low-risk policyholder, the expected claim is $59 \times 4 = 236$. The variance is $59^2 \times 4 = 13924$.

   The overall expected claim amount is

   $$0.16\times 592.857142857+0.68\times 266.66666666667+0.16\times 236 = 313.950476191$$

   For the variance, we can either calculate the raw moment, then subtract the square of the mean, or use the law of total variance.

   Calculating raw moments:

   The expected squared claim amount is

$$0.16 \times (592.857142857^2 + 2108877.55102) + 0.68 \times (266.66666666667^2 + 125811.965812) + 0.16 \times (236^2 + 13924) = 538704.035166$$

   The variance of the claim amount is therefore $538704.035166-313.950476191^2 = 440139.133665$.

   Law of total variance:

   The expected conditional variance is

   $$0.16\times 2108877.55102+0.68\times 125811.965812+0.16\times 13924 = 425200.384915$$

The variance of conditional expectation is

$$0.16\times 592.857142857^2+0.68\times 266.66666666667^2+0.16\times 236^2-313.950476191^2 = 14938.7487494$$

so the total variance is $425200.384915 + 14938.7487494 = 440139.133664$.

(b) Given that a policyholder's annual claims over the past 2 years were $1,421 and $119, what are the expectation and variance of the policyholder's claims next year?

- The likelihood of these claims for a high-risk policyholder is

$$\frac{2.4^2 \times 830^{2\times 2.4}}{(830+1421)^{3.4}(830+119)^{3.4}} = 1.78326350448 \times 10^{-7}$$

- The likelihood of these claims for a medium-risk policyholder is

$$\frac{4.6^2 \times 960^{2\times 4.6}}{(960+1421)^{5.6}(960+119)^{5.6}} = 7.3728993092 \times 10^{-8}$$

- The likelihood of these claims for a low-risk policyholder is

$$\frac{1421^3 \times 119^3 e^{-\frac{1421+119}{59}}}{59^8 \Gamma(4)^2} = 4.2216522276 \times 10^{-12}$$

The posterior probabilities are therefore:

$$\frac{0.16 \times 1.78326350448 \times 10^{-7}}{0.16 \times 1.78326350448 \times 10^{-7} + 0.68 \times 7.3728993092 \times 10^{-8} + 0.16 \times 4.2216522276 \times 10^{-12}} = 0.362688716863$$

$$\frac{0.68 \times 7.3728993092 \times 10^{-8}}{0.16 \times 1.78326350448 \times 10^{-7} + 0.68 \times 7.3728993092 \times 10^{-8} + 0.16 \times 4.2216522276 \times 10^{-12}} = 0.637302696937$$

and

$$\frac{0.16 \times 4.2216522276 \times 10^{-12}}{0.16 \times 1.78326350448 \times 10^{-7} + 0.68 \times 7.3728993092 \times 10^{-8} + 0.16 \times 4.2216522276 \times 10^{-12}} = 8.58619954718 \times 10^{-6}$$

This means that the overall expected claim amount is

$$0.362688716863 \times 592.857142857 + 0.637302696937 \times 266.6666666667 + 8.58619954718 \times 10^{-6} \times 236 = 384.972008619$$

The expected squared claim amount is

$$\begin{aligned}0.362688716863 \times (592.857142857^2 + 2108877.55102) + 0.637302696937 \times (266.666666667^2 + 125811.965812)\\ + 8.58619954718 \times 10^{-6} \times (236^2 + 13924) = 1017843.98095\end{aligned}$$

The variance of the claim amount is therefore $1017843.98095 - 384.972008619^2 = 869640.53353$.

2. *An insurance company sets the book pure premium for its auto insurance at \$630. The expected process variance is 17,215,000 and the variance of hypothetical means is 196,000. If a policyholder has aggregate claims of \$15,400 over the past 12 years, calculate the credibility premium for this policyholder's next year's insurance using the Bühlmann model.*

   The credibility of 12 years of experience is $Z = \frac{12}{12+\frac{17215000}{196000}} = 0.120202381561$. The credibility premium for this individual is therefore $0.120202381561 \times \frac{15400}{12} + 0.879797618439 \times 630 = \$708.53$.

3. *An insurance company has the following data on its auto insurance policies for a certain model of car.*

Year	*1*	*2*	*3*	*4*	*5*
Distance driven (1,000 km)	*6,221*	*4,495*	*7,251*	*6,304*	*7,554*
Aggregate claims	*\$1,484,100*	*\$1,226,000*	*\$1,609,300*	*\$1,355,300*	*\$1,664,700*

   The book premium is \$142 per 1000km. The variance of hypothetical means per 1000km is 842. The expected process variance per 1000km is 6,237,157,440. Using a Bühlmann-Straub model, calculate the credibility premium for Year 6 for a car that is expected to be driven 14,623km.

   The aggregate claims were of \$7,339,400 from 31,825,000km of driving. The credibility of 31,825,000km of driving is $Z = \frac{31825}{31825+\frac{6237157440}{842}} = 0.00427791289894$ Therefore the new premium per 1,000km is $0.00427791289894 \times \frac{7339400}{31825} + 0.995722087101 \times 142 = 142.379097686$. The total premium for a driver who expects to drive 14,623km is therefore $14.623 \times 142.379097686 = \$2,082.01$.

Standard Questions

4. *A workers' compensation insurer classifies workplaces as "low-risk" and "high-risk". It estimates that 84% of workplaces are low-risk. Annual claims from low-risk workplaces are modelled as following a Pareto distribution with $\alpha = 6.6$ and $\theta = 7585$. Annual claims from high-risk workplaces are modelled as following a Pareto distribution with unkown α and θ. A company is initially charged the book premium of \$1,192, then after claiming \$19,521 in its first year, it is charged a new premium of \$1,284. Which of the following is the value of the unknown parameter α, and what is the corresponding value of the unknown parameter θ?*

   (i) 1.94

   (ii) 2.77

(iii) 4.22

(iv) 6.50

The expected claim for low-risk workplaces is $\frac{7585}{5.6} = 1354.46428571$. Let x be the expected claim for high-risk workspaces. Since 16% of workspaces are high-risk, we have

$$\begin{aligned} 0.84 \times 1354.46428571 + 0.16x &= 1192 \\ 0.16x &= 54.25 \\ x &= 339.0625 \end{aligned}$$

Thus, the unknown parameters α and θ satisfy $\frac{\theta}{\alpha-1} = 339.0625$. After claiming $19,521 in its first year, the company is charged a new premium of $1,284. If p is the posterior probability of the company being high-risk, we have

$$\begin{aligned} (1-p) \times 1354.46428571 + p \times 339.0625 &= 1284 \\ 1015.40178571p &= 70.46428571 \\ p &= 0.0693954715283 \end{aligned}$$

The likelihood of claiming $19,521 if the company is low-risk is $\frac{6.6 \times 7585^{6.6}}{(7585+19521)^{7.6}} = 5.44444354718 \times 10^{-8}$. We let q be the likelihood of claiming $19,521 for a high-risk company. This q is given by $q = \frac{\alpha\theta^\alpha}{(\theta+19521)^{\alpha+1}}$. To get posterior probability 0.0693954715283, we must have

$$\begin{aligned} \frac{0.16q}{0.84 \times 5.44444354718 \times 10^{-8} + 0.16q} &= 0.0693954715283 \\ 0.16q &= 3.17368570819 \times 10^{-9} + 0.0111032754445q \\ 0.148896724556q &= 3.17368570819 \times 10^{-9} \\ q &= 2.1314677792 \times 10^{-8} \end{aligned}$$

Thus, we have two equations:

$$\begin{aligned} \frac{\theta}{\alpha-1} &= 339.0625 \\ \frac{\alpha\theta^\alpha}{(\theta+19521)^{\alpha+1}} &= 2.1314677792 \times 10^{-8} \end{aligned}$$

We therefore check the given values:

α	θ	$\frac{\alpha\theta^\alpha}{(\theta+19521)^{\alpha+1}}$
1.94	318.71875	$3.23339004565 \times 10^{-8}$
2.77	600.140625	$8.19351708421 \times 10^{-9}$
4.22	1091.78125	$8.44212529237 \times 10^{-10}$
6.50	1864.84375	$3.94585713177 \times 10^{-11}$

We see that (i) $\alpha = 1.94$ is the closest to the required value, and that the corresponding value of θ is 4561.

5. *An insurance company uses the Bühlmann-Straub model to calculate credibility. A new customer pays the book premium for 223 units of exposure, paying a total net premium of \$78,719 in its first year. It claims a total of \$92,218. In the second year, the customer pays a credibility premium of \$60,332 and claims a total of \$34,902. In the third year, the customer has 353 units of exposure, and pays a premium of \$121,666. How many units of exposure did the customer have in the second year?*

   In the first year, the book premium per unit of exposure is $\frac{78719}{223} = 353$, and the customer's average claim per unit of exposure is $\frac{92218}{223} = 413.533632287$. Let x be the number of units of exposure in the second year, and let $K = \frac{\text{EPV}}{\text{VHM}}$ for the credibility per unit of exposure. The credibility of 223 units of exposure is $Z = \frac{223}{223+K}$, and the credibility premium in the second year is $353(1-Z)+413.533632287Z = 353+60.533632287Z$ per unit of exposure. We therefore have the equation

   $$\left(353 + 60.533632287\frac{223}{223+K}\right)x = 60332$$

   The credibility of the first two units of experience is $Z_2 = \frac{223+x}{223+x+K}$, and the average losses per unit of experience over the first two years is $\frac{127120}{223+x}$. The credibility premium per unit of experience for the third year is therefore

   $$353(1-Z_2)+Z_2\frac{127120}{223+x} = 353\frac{K}{223+x+K}+\frac{127120}{223+x+K} = \frac{353K+127120}{223+x+K}$$

   Since the company pays \$121,666 for 353 units of experience, the credibility premium is $\frac{121666}{353} = 344.662889518$ per unit of experience. We therefore have the equation

   $$\frac{353K+127120}{223+x+K} = 344.662889518$$

   We thus just need to solve the two equations:

$$
\begin{aligned}
\left(353 + 60.533632287\frac{223}{223+K}\right)x &= 60332 \\
\frac{353K + 127120}{223 + x + K} &= 344.662889518 \\
(92218 + 353K)\,x &= 13454036 + 60332K \\
353K + 127120 &= 76859.8243625 + 344.662889518x + 344.662889518K \\
8.337110482K &= 344.662889518x - 50260.1756375 \\
K &= 41.3408086965x - 6028.48861677 \\
92218x + 353Kx &= 13454036 + 60332K \\
92218x + 353x(41.3408086965x - 6028.48861677) &= 13454036 + 60332(41.3408086965x - 6028.48861677) \\
14593.3054699x^2 - 2035838.48172x &= 2494173.67028x - 350256739.227x \\
14593.3054699x^2 - 4530012.152x + 350256739.227 &= 0 \\
x &= \frac{4530012.152 \pm \sqrt{4530012.152^2 - 4 \times 14593.3054699 \times 350256739.227}}{2 \times 14593.3054699} \\
&= 155.208570167 \pm 9.40783616037
\end{aligned}
$$

This gives $x = 164.616406327$. [Thus, $K = 41.3408086965 \times 164.616406327 - 6028.48861677 = 776.8867455$.]

[The other solution to the quadratic, $x = 145.800734007$ gives $K = 41.3408086965 \times 145.800734007 - 6028.48861677 = -0.96836438$, which is not possible.]

6. *A health insurance company is pricing its policies for an individual. It has 4 years of past history for this individual, and the annual claims from year i are denoted X_i. It uses the formula $\hat{X}_5 = \alpha_0 + \sum_{i=1}^{4} \alpha_i X_i$. It makes the following assumptions about the claims each year:*

   - *The expected aggregate claims is \$417 in each year.*
   - *The variance of the annual claims is 2,241,433.*
   - *The correlation between claims in Years i and j for $i \neq j$ is 0.55 if i and j are consecutive, and 0.40 otherwise.*

   Find a set of equations which can determine the values of α_i for $i = 0, 1, \ldots, 5$. [You do not need to solve these equations.]

   We use our standard equations:

$$
\begin{aligned}
\mathbb{E}(X_5) &= \alpha_0 + \sum_{i=1}^{4} \alpha_i \mathbb{E}(X_i) \\
\mathrm{Cov}(X_5, X_j) &= \sum_{i=1}^{4} \alpha_i \,\mathrm{Cov}(X_i, X_j)
\end{aligned}
$$

From the first condition, we have $\mathbb{E}(X_i) = 417$, $\mathrm{Var}(X_i) = 2241433$, and for $i \neq j$,

$$
\mathrm{Cov}(X_i, X_j) = \begin{cases} 0.55 \times 2241433 = 1232788.15 & \text{if } |i-j| = 1 \\ 0.4 \times 2241433 = 896573.2 & \text{otherwise} \end{cases}
$$

Substituting in the numbers given, these equations become:

$$
\begin{aligned}
417 &= \alpha_0 + 417\alpha_1 + 417\alpha_2 + 417\alpha_3 + 417\alpha_4 \\
896573.2 &= 2241433\alpha_1 + 1232788.15\alpha_2 + 896573.2\alpha_3 + 896573.2\alpha_4 \\
896573.2 &= 1232788.15\alpha_1 + 2241433\alpha_2 + 1232788.15\alpha_3 + 896573.2\alpha_4 \\
896573.2 &= 896573.2\alpha_1 + 1232788.15\alpha_2 + 2241433\alpha_3 + 1232788.15\alpha_4 \\
1232788.15 &= 896573.2\alpha_1 + 896573.2\alpha_2 + 1232788.15\alpha_3 + 2241433\alpha_4
\end{aligned}
$$

Dividing the covariance equations by 2241433, gives.

$$
\begin{aligned}
1 &= \frac{\alpha_0}{417} + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \\
0.4 &= \alpha_1 + 0.55\alpha_2 + 0.4\alpha_3 + 0.4\alpha_4 \\
0.4 &= 0.55\alpha_1 + \alpha_2 + 0.55\alpha_3 + 0.4\alpha_4 \\
0.4 &= 0.4\alpha_1 + 0.55\alpha_2 + \alpha_3 + 0.55\alpha_4 \\
0.55 &= 0.4\alpha_1 + 0.4\alpha_2 + 0.55\alpha_3 + \alpha_4
\end{aligned}
$$

[The solution to these equations is $\alpha_1 = 0.14470145$, $\alpha_2 = 0.13125336$, $\alpha_3 = 0.04034427$, $\alpha_4 = 0.41742873$ and $\alpha_0 = 111.03550323$.]