Modified Green's function for Laplacian

Consider the following problem:

$$(1) \quad \begin{cases} \Delta u = f(x), & x \in \Omega \\ \partial_n u = h(x), & x \in \partial\Omega \\ \int_\Omega u = 0 \end{cases}$$

<u>Q</u>: What is the corresponding Green's fcn ??

- If we try $\begin{cases} \Delta G = -\delta(x-x_0) \\ \partial_n G = 0 \end{cases}$ then integrating over Ω we get $\int_\Omega \Delta G = \int_\Omega \nabla \cdot \nabla G$

$$\Rightarrow \quad 0 = -\int_\Omega \delta = -1 \qquad = \int_{\partial\Omega} \underbrace{\partial_n G}_{0} = 0$$

$$0 = -1 \qquad \#$$

- Instead, we <u>try</u>:

$$\begin{cases} \Delta G + C = -\delta(x-x_0), & x \in \Omega \\ \partial_n G = 0, & x \in \partial\Omega \end{cases}$$

$$\Rightarrow \quad C \underbrace{\int 1}_{|\Omega|} = -1 \quad \Rightarrow \quad C = -\frac{1}{|\Omega|}$$

where $\quad |\Omega| = \int_\Omega 1 \, dx.$

So we define a Modified Green's Function to satisfy:

$$(MG)\quad \begin{cases} \Delta G(x,x_0) - \dfrac{1}{|\Omega|} = -\delta(x-x_0), & x \in \Omega \\ \partial_n G = 0, & x \in \partial\Omega. \end{cases}$$

Now recall that for any $u, v \in C^2(\Omega)$, we have:

$$\int_\Omega \Delta v\, u = \int_\Omega \Delta u\, v + \int_{\partial\Omega} (\partial_n v\, u - \partial_n u\, v)$$

Subbing in $v = G$, and u sol'n of (1), replacing Ω by $\Omega \setminus B_\varepsilon(x_0)$ and taking limit $\varepsilon \to 0$, we use the same argument as for Dirichlet B.C. to show that:

$$\boxed{u(x_0) = -\int_\Omega G(x,x_0) f(x)\,dx + \int_{\partial\Omega} h(x)\, G(x,x_0)\, dS(x)}$$

Modified Green's fcn on Rectangle :

We wish to solve (MG) on $\Omega = [0, L] \times [0, R]$.

Let's use Fourier series.

Write $\vec{x} = (x, y)$ $\quad \vec{x}_0 = (x_0, y_0)$.

Then $\delta(\vec{x} - \vec{x}_0) = \delta(x - x_0)\,\delta(y - y_0)$

Since we have Neumann B.C. at $x = 0, L$
$y = 0, R$,

we decompose in terms of $\cos\left(\frac{n\pi x}{L}\right)$.

So we write $\quad \delta(x - x_0) = \frac{a_0}{2} + \sum_{n \geq 1} a_n \cos\left(\frac{n\pi x}{L}\right)$;

$$a_n = \frac{2}{L} \int_0^L \delta(x - x_0) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L} \cos\left(\frac{n\pi x_0}{L}\right)$$

and $|\Omega| = LR$;

$$\Rightarrow \quad -\delta(x - x_0)\,\delta(y - y_0) = -\frac{\delta(y - y_0)}{L} - \sum \delta(y - y_0) \frac{2}{L} \cos\frac{n\pi x_0}{L} \cos\left(\frac{n\pi}{L}\right.$$

Next, write

$$G(x,y,x_0,y_0) = u(x,y) = u_0(y) + \sum_{n=1}^{\infty} u_n(y)\cos\left(\frac{n\pi x}{L}\right)$$

Then $\int_0^R u_0(y) \equiv 0$ since $\int u = 0$;

and $$u_n'' - \left(\frac{n\pi}{L}\right)^2 u_n = -\frac{2}{L}\cos\left(\frac{n\pi x_0}{L}\right)\delta(y-y_0),$$

and $$u_0'' - \frac{1}{LR} = -\frac{1}{L}\delta(y-y_0)$$

and $$u_n'(0) = u_n'(R) = 0, \quad n \geq 0.$$

To solve for u_n, first consider the following problem:

$$\begin{cases} V'' - \lambda^2 V = -\delta(y-y_0) \\ V'(0) = V'(R) = 0 \end{cases} \quad \left[\lambda = \frac{n\pi}{L}\right]$$

To satisfy BC, we write sol'n in the form:

$$V = \begin{cases} A\cosh(\lambda y)\cosh(\lambda(R-y_0)), & y < y_0 \\ A\cosh(\lambda y_0)\cosh(\lambda(R-y)), & y > y_0 \end{cases}$$

This also makes sure that V is continuous at $y=y_0$

To find A, we integrate $\int_{y_0^-}^{y_0^+}[V'' - \lambda^2 V = -\delta]$ over y_0 to get: $V'\Big|_{y_0^-}^{y_0^+} = -1$

$$\Rightarrow A\lambda\left(-\cosh \lambda y_0 \sinh(\lambda(R-y_0)) - \sinh \lambda_0 y_0 \cosh(\lambda(R-y_0))\right) =$$

<u>Now recall</u>: $\cosh x = \cos(ix)$
$\sinh x = -i \sin(ix)$

So that:

$$\begin{cases} \sinh(x \pm y) = \sinh x \cosh y \pm \sinh y \cosh x \\ \cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y \end{cases}$$

$$\Rightarrow A \lambda \sinh(\lambda R) = 1$$

$$\Rightarrow \quad V = \frac{1}{\lambda \sinh \lambda R} \begin{cases} \cosh \lambda y \cosh(\lambda(R-y_0)), & y < y_0 \\ \cosh \lambda y_0 \cosh(\lambda(R-y)), & y > y_0 \end{cases}$$

<u>OR</u>: using $\cosh A \cosh B = \frac{\cosh(A+B) + \cosh(A-B)}{2}$

we can write

$$\boxed{V = \frac{1}{2\lambda \sinh \lambda R}\left[\cosh \lambda(R - |y_0 - y|) + \cosh \lambda(R - (y+y_0))\right]}$$

Next, we solve:
$$\begin{cases} \omega'' - \frac{1}{R} = -\delta(y-y_0) \\ \omega'(0) = 0 = \omega'(R) \\ \int_0^R \omega = 0 \end{cases}$$

Imposing continuity of ω at $x = x_0$, we write:

$$\omega = \begin{cases} A + \dfrac{y^2 + (y_0 - R)^2}{2R}, & y < y_0 \\ A + \dfrac{y_0^2 + (y-R)^2}{2R}, & y > y_0 . \end{cases}$$

The jump condition $\omega' \Big|_{y_0^-}^{y_0^+} = -1$ is then automatically satisfied.

Using $\int_0^R \omega = 0$ we then get

$$A = -\frac{R}{6} .$$

Note that
$$\frac{y^2 + (y_0 - R)^2}{2R} = \frac{y_0^2 + y^2}{2R} - y_0 + \frac{R}{2}$$
$$\frac{y_0^2 + (y-R)^2}{2R} = \frac{y_0^2 + y^2}{2R} - y + \frac{R}{2}$$
$$\frac{y + y_0 + |y - y_0|}{2} = \begin{cases} y_0, & y < y_0 \\ y, & y > y_0 \end{cases}$$

So we get:
$$\boxed{\omega = \frac{R}{3} + \frac{y^2 + y_0^2}{2R} - \frac{1}{2}\left(y + y_0 + |y - y_0|\right)}$$

In summary, we obtain:

$$G(x, y, x_0, y_0) = u_0(y) + \sum_{n=1}^{\infty} u_n(y) \cos\left(\frac{n\pi x}{L}\right)$$

$$u_0(y) = \frac{R}{L3} + \frac{1}{L}\left\{\frac{y^2 + y_0^2}{2R} - \frac{1}{2}\left(y + y_0 + |y - y_0|\right)\right\}$$

$$u_n(y) = + \frac{\cos\left(\frac{n\pi x_0}{L}\right)}{\pi n \sinh\left(\frac{n\pi}{L} R\right)} \left[\cosh \frac{\pi n}{L}\left(R - |y_0 - y|\right) + \cosh\left(\frac{\pi n}{L} R - |y +\right.\right.$$

Resummation :

Recall that
$$G(\vec{x},\vec{x}_0) \sim \frac{-1}{2\pi} \ln |\vec{x}-\vec{x}_0| \text{ as } x \to x_0$$

For example

Now set
$$x_0 = \frac{L}{2}, \quad y_0 = \frac{R}{2}$$
$$x = \frac{L}{2} \qquad y = \frac{R}{2} + \delta \quad \text{and let } \delta \to 0.$$

We have:

$$f(\delta) = G\left(\frac{L}{2}, \frac{R}{2}+\delta, \frac{L}{2}, \frac{R}{2}\right) = \frac{R}{12L} - \frac{1}{2L}|\delta| + O(\delta^2)$$
$$+ \sum_{n=1}^{\infty} \cos^2\left(\frac{n\pi}{2}\right) \frac{\left(\cosh \frac{\pi n (R-|\delta|)}{L} + \cosh\left(\frac{\pi n \delta}{L}\right)\right)}{n\pi \sinh \frac{n\pi R}{L}}$$

Question Where is the log???

I.E. we Know that $f(\delta) \sim -\frac{1}{2\pi} \ln|\delta| + R_0(\delta)$
as $\delta \to 0$. Can we compute $R(\delta)$??

$$\left(\cos\frac{n\pi}{2}\right)^2 = (-1)^n + 1 = \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

so we rewrite the sum as

$$S = \sum_{n=1}^{\infty} \frac{\cosh\left(\frac{2\pi n}{L}(R - |\delta|)\right) + \cosh\left(\frac{2\pi n \delta}{L}\right)}{2\pi n \sinh\left(2\pi n \frac{R}{L}\right)}$$

So we consider first a sum of the type

$$\sum_{1}^{\infty} \frac{\cosh(na)}{n \sinh(nb)}, \quad |a| < b.$$

Now

$$\frac{\cosh na}{\cosh nb} = \frac{e^{na} + e^{-na}}{e^{nb} - e^{-nb}} = \frac{e^{n(a-b)} + e^{n(-a-b)}}{1 - e^{-2nb}}$$

$$= \left(e^{n(a-b)} + e^{n(-a-b)}\right)\left(\sum_{m=0}^{\infty} e^{-2nbm}\right)$$

Thus

$$\sum \frac{\cosh na}{n \sinh nb} = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{\left(e^{-2bm + a - b}\right)^n + \left(e^{-2bm - a -}\right.}{n}$$

Assuming $|a| < b$, the partial series are <u>absolutely convergent</u> so we can interchange the summation signs so that

$$\ldots = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty}$$

Now $\displaystyle\sum_{n=1}^{\infty} \frac{r^n}{n} = -\ln(1-r)$, $|r|<1$

so we get

$$\sum \frac{\cosh na}{n \sinh nb} = \sum_{m=0}^{\infty} -\ln\left(1-e^{-2bm+a-b}\right) - \ln\left(1-e^{-2bm-a}\right.$$

$$= -\ln \prod_{m=0}^{\infty}\left(1-e^{-2mb-b}\left(2\cosh a + e^{-2bm-b}\right)\right)$$

$$= -\ln \prod \left(1-e^{-2mb+a-b}\right)\left(1-e^{-2mb-a-b}\right)$$

Back to $\displaystyle S = \frac{1}{2\pi} \sum \frac{\cosh na_1}{n \sinh nb} + \frac{\cosh na_2}{n \sinh b}$

<u>where</u> $b = \dfrac{2\pi R}{L}$; $a_1 = b - \dfrac{2\pi}{L}|\delta|$

$a_2 = \dfrac{2\pi}{L}\delta$

$b - a_1 = \dfrac{2\pi}{L}|\delta|$

$b + a_1 \sim 2b$

$b \pm a_2 \sim b$

$$\Rightarrow 2\pi S = -\ln \prod_{m=0}^{\infty}\left(1-e^{-2mb-b}\right)^2\left(1-e^{-2mb-2b}\right)\left(1-e^{-2mb-}\right.$$

$$\sim -\ln|\delta| - \ln\frac{2\pi}{L} - 2\ln\prod_{m=1}^{\infty}\left(1-e^{-mb}\right)$$

So $$R_0 = \frac{R}{12L} - \frac{1}{2\pi}\ln\frac{2\pi}{L} - \frac{1}{\pi}\ln\prod_{m=1}^{\infty}\left(1-e^{-2\pi\frac{R}{L}}\right)$$

Where $$R_0 = \lim_{(x,y)\to(x_0,y_0)} G + \frac{1}{2\pi}\ln|(x_0,y_0)-(x,y)| \Bigg|_{\substack{x_0=\frac{L}{2}\\ y_0=\frac{R}{2}}}.$$

Example:

- Take $R=0.5$, $L=2$ then partial product is

$M=1$: $R_0 = -0.08717$

$M=2$: $R_0 = -0.073$

3 $\quad -0.070$

5 $\quad -0.06952$

10 $\quad -0.069491169925$

- Take $R=2$, $L=0.5$

$M=1$: $R_0 = -0.0694916866$

$M=2$: $R_0 = -0.0694916866$

Interesting identity: Since R_0 is unchanged if we switch R, L:

$$\boxed{\prod_{m=1}^{\infty}\frac{(1-e^{-2m\pi\rho})}{(1-e^{-2m\pi\rho^{-1}})} = \rho^{-\frac{1}{2}}e^{\frac{\pi}{12}(\rho-\frac{1}{\rho})}, \quad \rho>0}$$

References

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