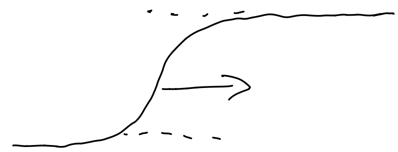
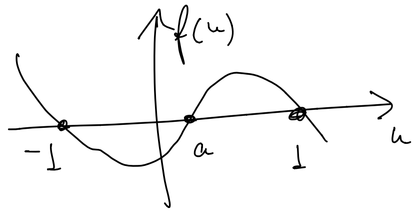
"Wave" propagation

$$u_t = u_{yy} + f(u)$$



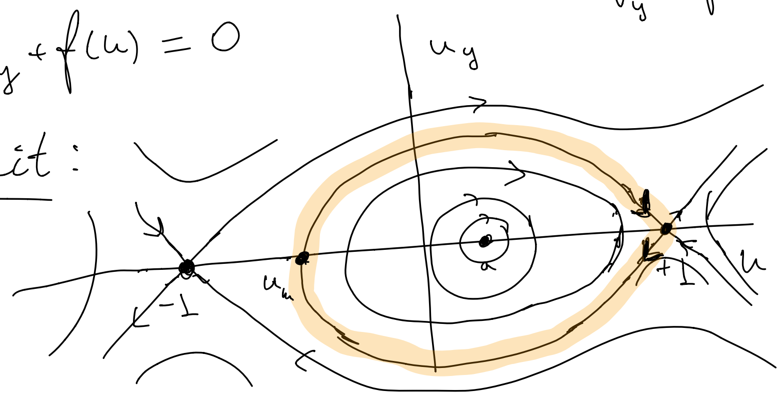
$$\cdot\ f(u) = -(u-1)(u+1)(u-a)$$



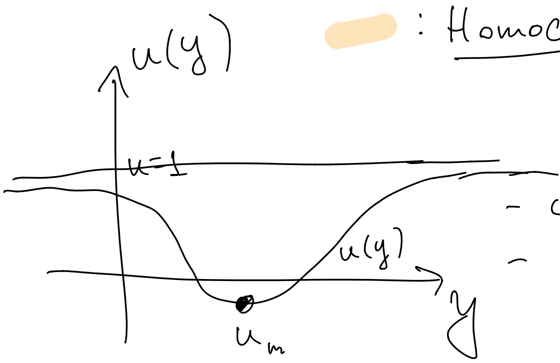
$u_y = v$
$v_y = -f(u)$

S.S. $u_t = 0$ $\quad u_{yy} + f(u) = 0$

Phase portrait:



Case $a > 0$
$a < 1$



: Homoclinic orbit connects stable & unstable manifolds of same saddle point

- $\overset{y}{\text{connects}}$ +1 to +1
- goes through $u = u_m$, $u_y = 0$

$\nearrow$ minimum.

Compute homoclinic orbit:

$$u_y u_{yy} + u_y f(u) = 0$$

$$\iff \left(\frac{1}{2}u_y^2\right) + F(u) = C$$

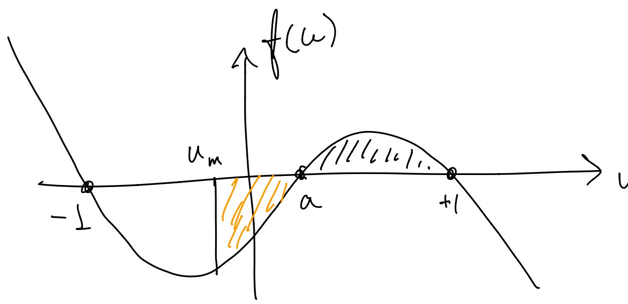
$$F(u) = \int_1^u f(u)\,du$$

$$F'(u) = f(u)$$

$C = 0$ [plug in $u=1$, $u_y = 0$]

$$\Rightarrow \frac{u_y^2}{2} + F(u) = 0 \qquad u = u_m,\ u_y = 0$$

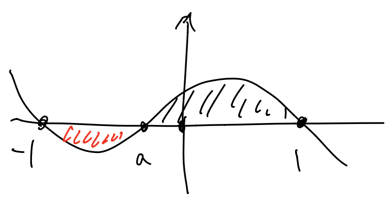
$$\Rightarrow F(u_m) = 0 \Rightarrow \boxed{\int_1^{u_m} f(u)\,du = 0}$$



determines u_m.

Note a must be ≥ 0 to have $\int_{u_m}^1 f = 0$

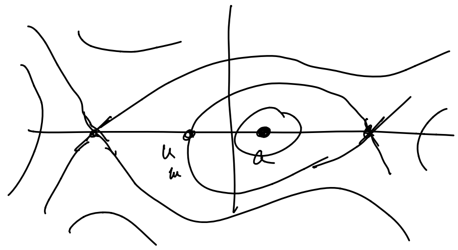
If $a < 0$:



$\Rightarrow$ No solution to $\int_{u_m}^1 f = 0$

Phase portrait :

$a > 0$

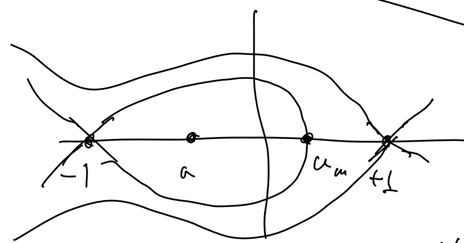


Homoclin. orbit connecting (+1) to (+1)

"fish swimming left"

$$\int_{u_m}^{1} f(u)\,du = 0$$

$a < 0$

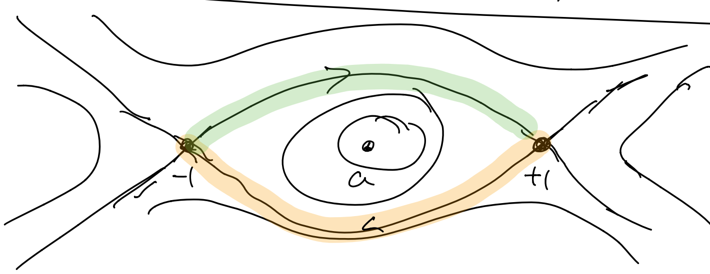


$$\int_{-1}^{u_m} f(u)\,du = 0$$

Homoclinic orbit connecting (-1) to (-1)

(fish swimming "right")

$a = 0$

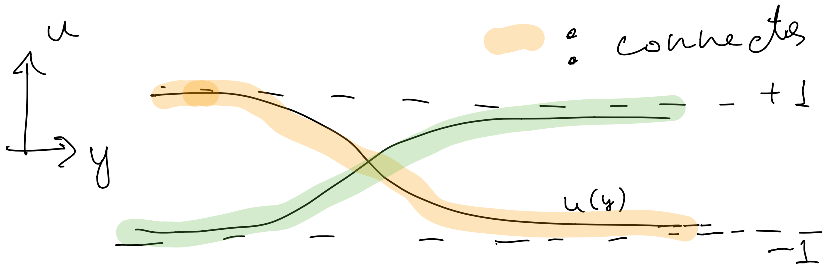


(mutant fish)

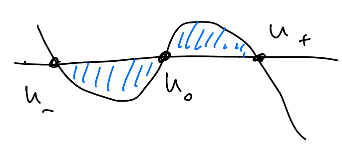
: <u>Heteroclinic orbits</u>

: connects -1 to $+1$

: connects $+1$ to -1



Condition for having heteroclinic orbit

- $a = 0$
- General $f(u) =$ 

- $f(u_\pm) = 0, \quad f(u_0) = 0$

- $$\boxed{\int_{u_-}^{u_+} f(u)\, du = 0}$$

"Equal-area condition".

Suppose $f(u) = -2(u-1)(u+1)(u-a)$

- If $a=0$, sol'n to $u_{yy} + 2u - 2u^3 = 0$
  is given by $u(y) = \pm \tanh(y)$

- then $u_t = u_{yy} + 2u - 2u^3$
  - has s.s. $u(y,t) = \tanh(y)$
  - <u>marginally stable</u>
    [zero eigenvalue corresponding to translation invariance]

- Suppose $a \equiv \varepsilon$ <u>small</u>, <u>non-zero</u>
  - then you get slowly travelling "wave" of form $u \approx \tanh(y - ct)$
    $c \equiv$ speed to be found.
    $c \equiv$ small if a small.

<u>To find c:</u>

<u>General eq'n:</u>

$$u_t = u_{yy} + f(u) + \varepsilon g(u)$$

Where $f(u)$ is well-balanced nonlinearity satisfying equal-area condition:

$$\int_{u_-}^{u_+} f(u)\, du = 0$$

$$f(u_\pm) = 0$$

- $g(u)$ some "perturbation"
- $\varepsilon \ll 1$

Expand: $u = U_0(z) + \varepsilon U_1(y,t)$

where $z = y - \varepsilon c t$

- $U_t = -\varepsilon c U_{0z} + \ldots$

$O(1)$: $U_{0zz} + f(U_0) = 0$

$O(\varepsilon)$: $-c U_{0z} = U_{1zz} + f'(U_0) U_1 + g(U_0)$ (*)

$U_0 \equiv$ heteroclinic connection

Multiply (*) by U_{0z} & integrate:

$$-c \int U_{0z}^2 = \int \left(U_{1zz} + f'(U_0) U_1\right) U_{0z} + \int g(U_0) U_{0z}$$

Note: int. by parts twice

$$\int u_{1zz}u_{0z} = \int u_{0zzz}u_1$$

[since boundary terms $\to 0$]

$$\Rightarrow \int (u_{1zz} + f'(u_0)u_1)u_{0z}$$

$$= \int u_1 \underbrace{(u_{0zzz} + f'(u_0)u_{0z})}_{0} = 0$$

$$\Rightarrow \boxed{-c\int u_{0z}^2 = \int g(u_0)u_{0z}}$$

Example: $u_t = u_{yy} - 2(u-\varepsilon)(u-1)(u+1)$

$a \approx \varepsilon$

$\Rightarrow f(u) = -2u(u-1)(u+1)$,

$u_0(z) = \tanh(z)$ $\quad g(u) = 2(u-1)(u+1)$,

$u_t = u_{yy} + f(u) + \varepsilon g(u)$

Then $u_0 = \tanh(y - \varepsilon c t)$

where $c = -\dfrac{\int g(u_0)u_{0z}}{\int u_{0z}^2}$

$$g(u_0)\, u_{0_z} = (2u^2 - 2)\, u_z = \frac{d}{dz}\left(\frac{2}{3}u^3 - 2u\right)$$

$$\Rightarrow \int g(u_0)\, u_{0_z}\, dz = \left(\frac{2}{3}u^3 - 2u\right)\Big|_{-\infty}^{\infty}$$

$$= \left(\frac{2}{3} - 2\right) \cdot 2 = -\frac{8}{3}$$

and $\displaystyle \int u_{0_z}^2 = \int_{-\infty}^{\infty} \operatorname{sech}^4 z\, dz = \frac{4}{3}$

$$\Rightarrow \boxed{c = 2}$$

$$\Rightarrow \boxed{u \sim \tanh(y - 2\varepsilon t)}$$