MATH 5230/4230, Homework 4

1. Consider the eigenvalue problem

$$\begin{aligned} \Delta u + \lambda u &= 0, \quad \text{inside } B_1(0) \subset \mathbb{R}^2 \\ \partial_n u &= 0, \quad \text{on } \partial B_1(0) \\ u &= 0, \quad \text{on } \partial B_\varepsilon(\xi) \end{aligned} \tag{1}$$

Here, ε is assumed small and $B_\varepsilon(\xi) \subset B_1(0)$.

(a) First, suppose that $\xi = 0$ so that the problem is radially symmetric. In this case, find an implicit expression for λ in terms of Bessel J_0 and Y_0 functions. See Wikipedia or ask me if you need more info on Bessel functions.

(b) Using the following expansions for Bessel J_0 and Y_0 functions of small arguments,

$$Y_0(z) \sim \frac{2}{\pi} \ln(z) + \frac{2}{\pi}(\gamma - \ln 2) \text{ as } z \to 0,$$
$$J_0(z) \sim 1 + O(z^2) \text{ as } z \to 0,$$

where $\gamma = 0.577...$ is the Euler constant, find the asymptotic formula for λ in the limit $\varepsilon \to 0$.

(c) Use Maple to compute λ as determined by (a). Hint: the command `fsolve` will be useful here: for example `fsolve(x^2=2,x=1.4);` uses fsolve to solve for $\sqrt{2}$, the second argument provides an initial guess. Then compare with the asymptotic formula for λ that you obtained in part (b). Do this for two values, $\varepsilon = 0.1$ and $\varepsilon = 0.05$. Comment on what you observe for the error behaviour.

(d) Now do the case of general ξ in (1). Here are some steps:

- Assume λ is small and expand $u(x) = 1 + \lambda u_1(x) + \ldots$.
- In the outer region (away from ξ), $u_1 \sim AG(x, \xi)$ where A is some constant that you will need to determine and G is the same Neumann's Green's function that we saw in class.
- In the inner region, change variables $x = \xi + \varepsilon y$ and solve.
- Match inner and outer region to estimate λ.

(e) Double-check that your answer in (d) agrees with part (b) when $\xi = 0$. How does the answer depend on ξ?

(f) Suppose that you replace $B_\varepsilon(\xi)$ by an ellipse $E_\varepsilon(\xi)$ that is parallel with x and y axes and whose major and minor axes have radii εa and εb, respectively. Determine how a and b effects λ.