MATH 5230/4230, Homework 5

1.

(a) Solve the PDE $u_t + xu_x = 0$ subject to initial condition $u(x, 0) = x$.

(b) Solve the PDE $u_t + xu_x = -u$ subject to initial condition $u(x, 0) = x$.

2. Consider the system

$$u_t + (2 - u)u_x = 0; \qquad u(x, 0) = 1 + \tanh(x).$$

(a) Determine the characteristic curves for this ODE.

(b) Show that the solution develops a shock and compute the time $t = t_s$ at which the shock first occurs, as well as the location $x = x_{shock}$ at which the shock takes place.

(c) Sketch the solution profile $u(x, t)$ for $t = 0, \ 0.5t_s, \ t_s, 1.5t_s$.

3.

(a) Consider the system

$$u_t + uu_x = 0; \quad u(x, 0) = \phi(x).$$

where ϕ is is given by

$$\phi(x) = \begin{cases} 1, & x < 0 \\ 1 - x, & 0 < x < 1 \\ 0, & x > 0. \end{cases} \quad . \tag{1}$$

The initial condition develops a shock at some time $t = t_s$. Compute the value of t_s and sketch the solution at $t = 0, t = t_s/2$ and $t = t_s$.

(b) Replace the PDE by $u_t + uu_x = -u$, but keep same initial conditions. Solve for characteristics and sketch solutions for several values of t. Show that there is no shock.

(c) Finally, repeat part (a) but for $u_t + uu_x = u$. Solve for characteristics and sketch the solution for several values of t. Show that there is a shock and compute t_{shock}.

4. Consider the system

$$u_t + uu_x - u = \varepsilon u_{xx}; \quad u(x, 0) = \phi(x).$$

with initial condition given by

$$\phi = \begin{cases} 1, x < 0 \\ 0, x > 0 \end{cases} .$$

(a) This system forms a shock at $t = 0$, which is smoothed out by the diffusion term for $t > 0$. Compute the evolution equation for the location of the shock $x = s(t)$.

(b) Use FlexPDE to verify what you found in part (a) numerically. That is, you should hand in plots that compare the location of the interface, $s(t)$ computed numerically, with your asymptotic predictions. See website for a sample for Burger equation.