Maximizing network diffusivity subject to resource constraints

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Algebraic Connectivity (AC) is a measure of how fast the information diffuses through a graph. - corresponds to 2nd eigenvalue of Laplacian matrix of graph.

Q: Maximize AC among all graphs of given # vertices n, and given # edges m.

- Too hard, how about:

Q': Further restrain to d-regular graphs only $\left(m = \frac{dn}{2}\right)$

- Still hard

Q'': Further restrict to d-regular and with fixed diameter D, or fixed girth g.

- Alon-Boppana-Friedman bound:

- $AC \leq d - 2\sqrt{d-1}\cos\frac{\pi}{\lfloor D/2 \rfloor}$

- Note that $D \sim C \log n \to \infty$ as $n \to \infty$, so that $d - 2\sqrt{d-1}\cos\frac{\pi}{\lfloor D/2 \rfloor} \to d - 2\sqrt{d-1}$ as $n \to \infty$

- on average, $AC \sim d - 2\sqrt{d-1}$ as $n \to \infty$ (with fixed d)

Tighter bound:

Theorem 1. Suppose that a d–regular graph has girth g and diameter D and let AC be its algebraic connectivity. Then

$$AC \leq d - 2(d-1)^{1/2}\cos\theta$$

where θ is two of the following four values, depending on the parity of D and g.

- If D is even with $D = 2K$, then θ is the smallest positive root of

  ← prev known

$$\tan(\theta K) = -\frac{d}{d-2}\tan\theta.$$

- If D is odd with $D = 2K - 1$, then θ is the smallest positive root of

  New

$$\tan(\theta K) = -\frac{\left(2\sqrt{d-1}\cos\theta + d\right)\sin\theta}{\sqrt{d-1}\left(d - 2\cos^2\theta\right) + (d-2)\cos\theta}.$$

- If g is even with $g = 2K$, then $\theta = \pi/K$.
- If g is odd with $g = 2K + 1$, then θ is the smallest root of

$$\tan(\theta K) = -\frac{\sin\theta}{(d-1)^{-1/2} + \cos\theta}.$$

Maximal graphs: those that attain the above upper bound.

Upper bound for AC in terms of girth									
g \ d	3	4	5	6	7	8	9	10	11
3	**4.0000**	**5.0000**	**6.0000**	**7.0000**	**8.0000**	**9.0000**	**10.000**	**11.000**	**12.000**
4	**3.0000**	**4.0000**	**5.0000**	**6.0000**	**7.0000**	**8.0000**	**9.0000**	**10.000**	**11.000**
5	**2.0000**	*2.6972*	*3.4384*	*4.2087*	*5.0000*	*5.8074*	*6.6277*	*7.4586*	*8.2984*
6	**1.5858**	**2.2679**	**3.0000**	**3.7639**	*4.5505*	**5.3542**	**6.1716**	**7.0000**	7.8377
7	*1.1864*	*1.7466*	*2.3738*	*3.0443*	*3.7458*	*4.4709*	*5.2147*	*5.9739*	*6.7460*
8	**1.0000**	1.5505	2.1716	2.8377	3.5359	4.2583	5.0000	5.7574	6.5279
9	*0.8088*	*1.3004*	*1.8706*	*2.4913*	*3.1481*	*3.8322*	*4.5380*	*5.2616*	*6.0000*
10	*0.7118*	*1.1975*	*1.7639*	*2.3820*	*3.0366*	*3.7191*	*4.4235*	*5.1459*	*5.8833*
11	*0.6069*	*1.0600*	*1.5983*	*2.1912*	*2.8229*	*3.4840*	*4.1685*	*4.8721*	*5.5916*
12	**0.5505**	1.0000	1.5359	2.1270	2.7574	3.4174	4.1010	4.8038	5.5228
13	*0.4872*	*0.9168*	*1.4356*	*2.0114*	*2.6277*	*3.2748*	*3.9462*	*4.6376*	*5.3456*

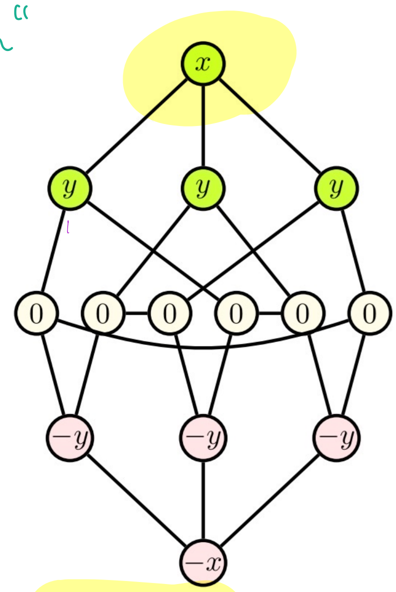
Upper bound for AC in terms of diameter									
D \ d	3	4	5	6	7	8	9	10	11
3	**2.0000**	**3.0000**	**4.0000**	**5.0000**	**6.0000**	**7.0000**	**8.0000**	**9.0000**	**10.000**
4	**1.2679**	**2.0000**	**2.7639**	**3.5505**	**4.3542**	**5.1716**	**6.0000**	6.8377	**7.6834**
5	**1.0000**	**1.6972**	2.4384	3.2087	4.0000	4.8074	5.6277	6.4586	7.2984
6	**0.7639**	1.3542	2.0000	2.6834	3.3944	4.1270	4.8769	5.6411	6.4174
7	**0.6571**	1.2266	1.8587	2.5321	3.2356	3.9621	4.7070	5.4671	6.2398
8	**0.5505**	1.0665	1.6508	2.2810	2.9446	3.6340	4.3440	5.0709	5.8123
9	**0.4965**	1.0000	1.5762	2.2006	2.8597	3.5456	4.2527	4.9772	5.7164
10	0.4384	0.9111	1.4601	2.0598	2.6964	3.3613	4.0487	4.7546	5.4762
11	0.4069	0.8717	1.4156	2.0118	2.6456	3.3083	3.9939	4.6984	5.4187
12	0.3714	0.8167	1.3436	1.9245	2.5444	3.1941	3.8677	4.5607	5.2701
13	0.3512	0.7912	1.3148	1.8934	2.5115	3.1599	3.8323	4.5243	5.2329

TABLE 1 Upper bounds for AC in terms of girth and diameter. Known attainable bounds are in bold. Known unattainab are in italics. The rest are unknown.

- Many of the diameter bounds are attainable...

- Ex and main idea:

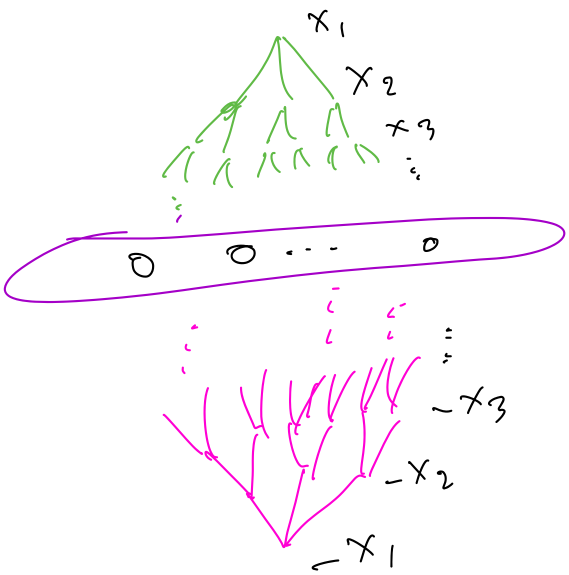
"H3 graph"



  - $D=4,\ d=3,\ n=14$
  - Double-tree structure
  - Eigenvector structure as shown:
    - $\lambda x = 3x - 3y$
    - $\lambda y = 3y - x$

  $\Rightarrow \lambda$ eig of $\begin{bmatrix} 3 & -3 \\ -1 & 3 \end{bmatrix}$: $\lambda = 3 - \sqrt{3} = 1.2679$

- Generalize to any even $D = 2K$



$$\lambda \begin{pmatrix} x_1 \\ \vdots \\ x_K \end{pmatrix} = \begin{pmatrix} 3 & -3 & & & & \\ -1 & 3 & -2 & & 0 & \\ & -1 & 3 & -2 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & -1 & 3 & -2 \\ & & & & -1 & 3 \end{pmatrix}$$

Eig. given by

$$\lambda = 3 - 2\sqrt{3}\cos\theta,$$
$$\tan(\theta K) = 3\tan\theta$$

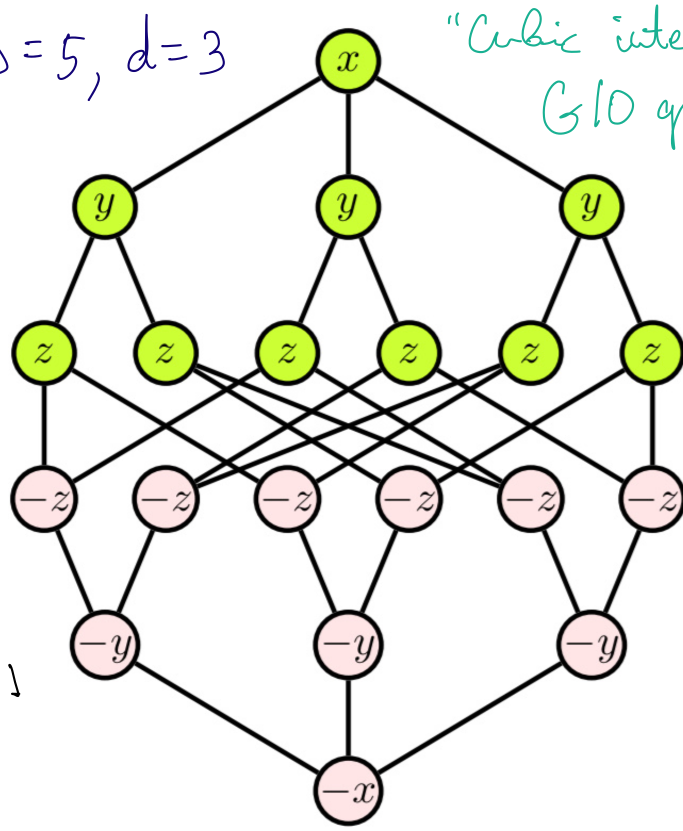
Thm [Alon-Boppana-Friedman]:

- This actually gives the upper bound, regardless if graph has double tree structure etc
- Proof uses Rayleigh quotient...

Odd diameter:

Ex $D=5$, $d=3$

"Cubic integral G10 graph"



$$\begin{cases} \lambda x = 3x - 3y \\ \lambda y = 3y - 2z - x \\ \lambda z = 3z - y - (-2z) \end{cases}$$

$$\lambda : \begin{bmatrix} 3 & -3 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 5 \end{bmatrix} \rightsquigarrow \lambda_{min} = 1$$

Thm: This vertex assignment gives upper bound:

$D = 2K-1$, $(d=3)$:

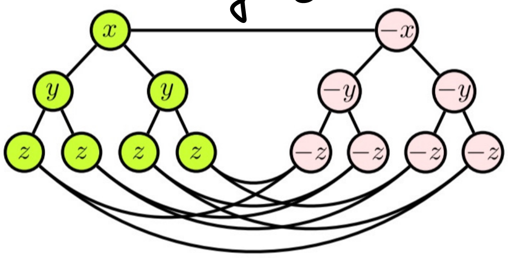
$$\begin{cases} \lambda = 3 - 2\sqrt{3}\cos\theta, \\ \tan(\theta K) = \dfrac{(2\sqrt{2}\cos\theta + 3)\sin\theta}{\sqrt{2}(3 - 2\cos^2\theta) + \cos\theta} \end{cases}$$

- Improvement over A-B-F bound in the case of odd D
- Proof is abit more involved because the two trees are not separated. Additional argument is required to "separate" them

girth Bound:

g = 6 (even)

(Heawood graph)



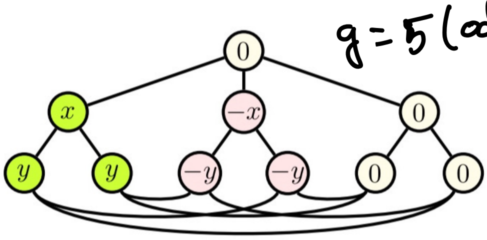
(c)

$$\lambda : \begin{bmatrix} 4 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 5 \end{bmatrix}$$

$(\lambda = 1.585)$

g = 5 (odd)

(Petersen graph)



(d)

$$\lambda : \begin{bmatrix} 3 & -2 \\ -1 & 5 \end{bmatrix}$$

$(\lambda = 2)$

Additional constraints:

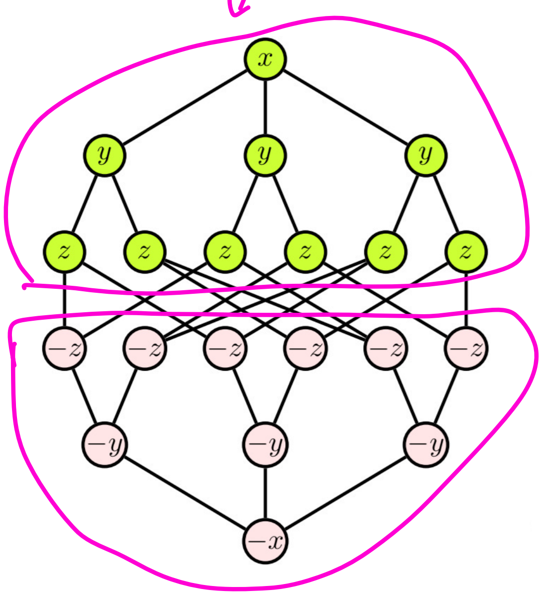
Thm: Girth-maximal graphs must be <u>Moore graphs</u>. (Very restrictive, only exists if $g = 3, 4, 6, 8, 12$ or $d = 3$ and $g = 5$)

Thm: If $D = 2K-1$ is <u>odd</u>, then a D-maximal graph must be bipartite, and consist of two disjoint Moore trees of K levels, that is $n = 2\left(1 + d + d(d-1) + \dots d(d-1)^{K-2}\right)$.

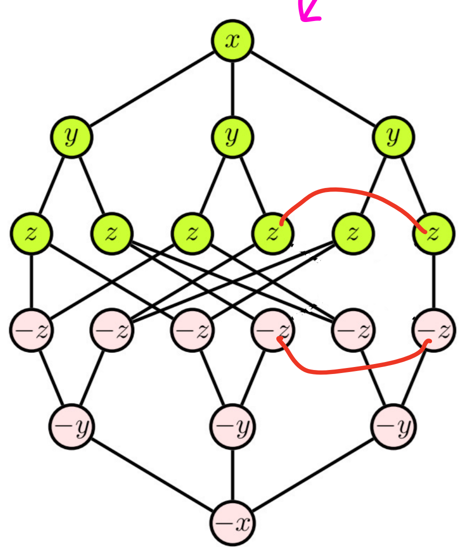
Moreover all edges from leafs of one tree must go to the other tree.

<u>Ex</u>:

(possibly) maximal



<u>not</u> maximal



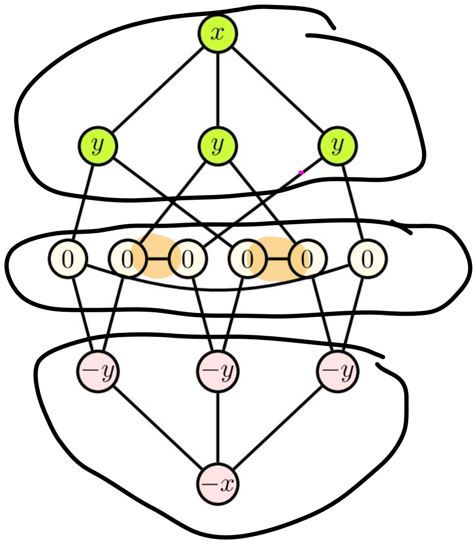
Thm: if $D = 2K$ is even, then a D-maximal graph consists of two disjoint Moore trees of K levels, plus a center that is not part of either trees; and all edges from leafs of both trees must go into center. Such graph has

$$n \geq \underbrace{2\left(1 + d + d(d-1) + \dots + d(d-1)^{K-2}\right)}_{\text{2 trees}} + \underbrace{2(d-1)^{K-1}}_{\text{center vertices}}$$

inequality

center vertices (at least this number)

Examples:



$D = 4$, $n = 14$
$d = 3$

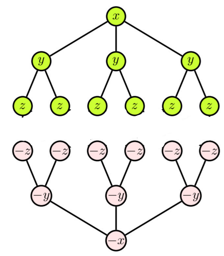
center, note edges inside center possible

- $n_{min} = 12$ but no maximal such graphs.

- Note that "$n = \dots$" for D odd, but "$n \geq \dots$" for even D

Algorithm:

- For odd D, initial configuration is double tree:



(ex $D=5$, $d=3$)

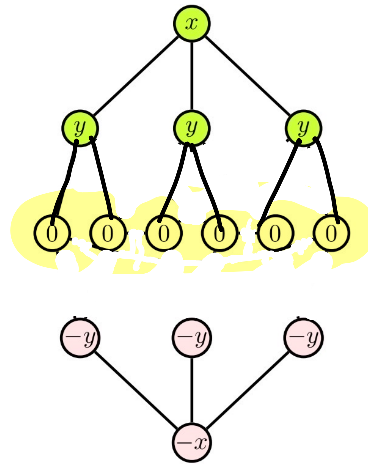
Loop:

- Choose a random edge to add, subject to the following constraints:
    - must keep degree $\leq d$
    - must keep girth $\geq g$ [choose $g = D+1$]
- If cannot add any more:
    - If the graph is completed [every vertex has deg. d],
        - compute its AC
        - If AC = max bound → Done!
          Else reset & try again
    - Else delete a few random edges

endloop

Even D: Similar, but initial config. is double tree with a center, ex:

- Can add edges between white & red or white & white



Computational Results:

$d=3$ summary

D	AC	n	#graphs	Comments
3	2	8	**1**	The 3-cube, see §6.2.
4	1.2679	14	**1**	Graph 3H (see Figure 1(a))
		16	**1**	Möbius Kantor Graph
		18	**1**	Pappus graph
5	1	20	**5**	All have girth 6; includes the Desargues graph
6	0.7639	32	**2**	Both have girth 7
		34	**2**	Both have girth 7, cospectral
		36	**2**	Both have girth 8.
		38	**1**	Girth 8
		40	**2**	Both have girth 8
		42	**2**	Both have girth 8, cospectral
7	0.6571	44	**45**	All have girth 8.
8	0.5505	68	12	Two of girth 8 and ten of girth 9
		80	1	Girth 10
		90	3	Girth 10
9	0.4965	92	481	All have girth 10
10	0.4384	≥ 140?	???	Open question!! Maybe $g=11$?

- $d=3$, $D=10$ is open!
- $D=8$: what about $n=70, 72, 74 \dots 88$?

Complexity:

- $D=7$: All 45 graphs were found in minutes (under an hour)
- To find $D=9$ we used 6 cpu cores running several days. This resulted in about 1500 maximal graphs. Of these, 481 were non-isomorphic.

d = 4 summary:

D	n	# graphs	comments
3 (AC=3)	10	**1**	See §6.1.
4 (AC=2)	16	**6**	Girth 4; group sizes: 6,8,12,32,48 (Hoffman), 384 (4
	17	**0**	Girth 4: 193900 graphs, none maximal
	19-21	**0**	Girth 5
	22	**3**	Girth 5, group sizes 2, 4, 8
	23	**2**	Girth 5; group sizes 1, 4
	24	**2**	Girth 5; co-spectral, group sizes 16, 16
	28	**1**	Unique graph of girth 6 on 28 vertices
	30	**1**	Girth 6, see §6.4 and Figure 9
	32	**1**	Girth 6

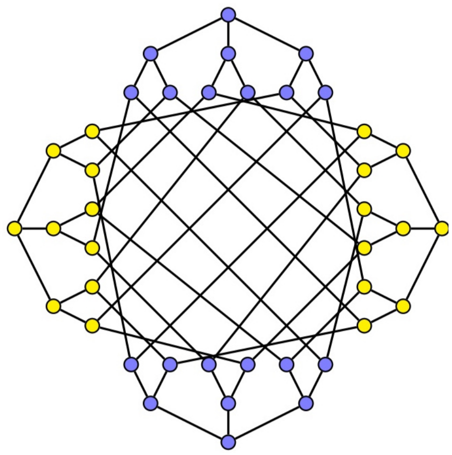
Gallery of maximal D-graphs

$d=3,\ D=6$

$n=40,\ g=6$

$|Aut|=480$

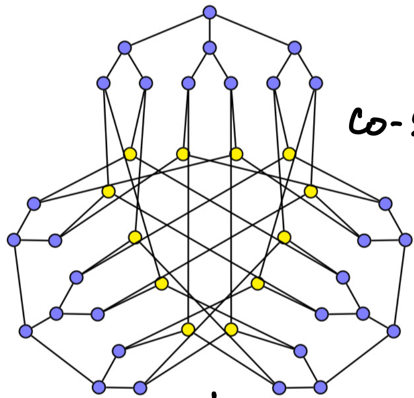
$AC=0.7639$



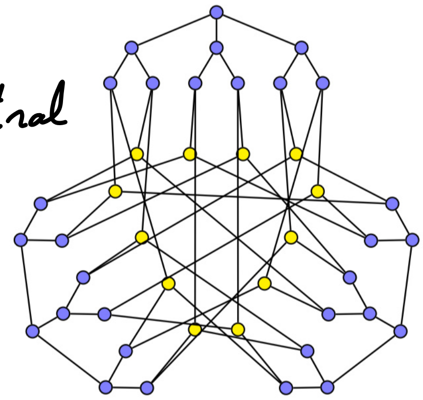
$d=3,\ D=6$

$n=42,\ g=6$

$AC=0.7639$



co-spectral



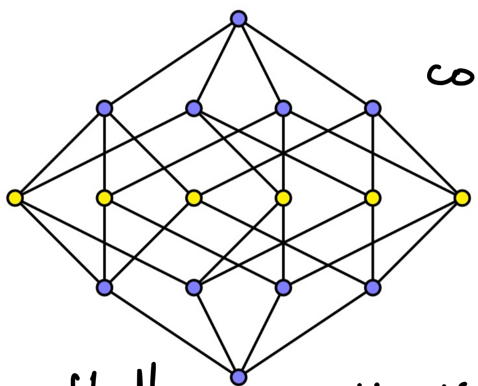
$|aut|=48$

$|Aut|=24$

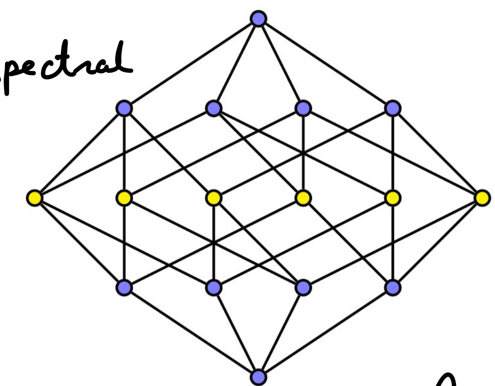
$d=4, D=4$

$n=16,\ g=4$

$AC=2$



co-spectral



Hoffman, $|Aut|=48$

Hypercube

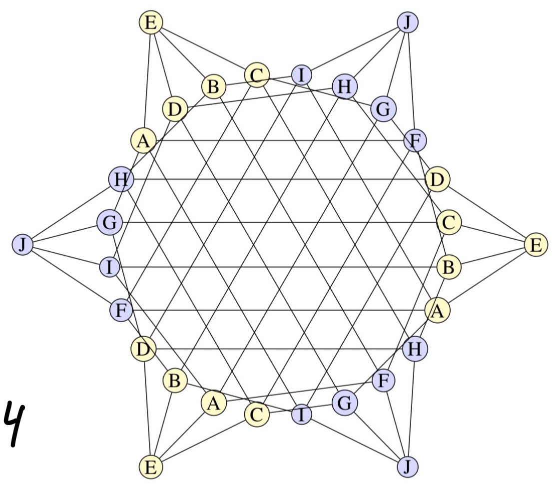
$|Aut|=384$

$d=4, D=4$

$n=30,\ AC=2$

$|Aut|=720$

(part of $d=p^{\alpha}, D=4$ family)



D-maximal family with D=4, d=prime power

Here, we will construct a d-regular graph G which is an incidence graph of a subset of $PG(2,d)$ (with d a prime power). Its order is $2d^2-2$. This graph is likely to be the same as the girth-6 graph of the same order from [34,35,36], although we use a different construction here to compute its spectrum and girth.

Consider the subset of lines and points of $PG(2,d)$ of the form $(1,b,c)$, where one of b,c are non-zero. For example when $d=3$, there are 8 such lines and points, namely:

$$(1,0,1),\ (1,0,2);\quad (1,1,0),\ (1,2,0);\quad (1,1,1),\ (1,2,2);\quad (1,1,2),\ (1,2,1). \tag{29}$$

It is easy to see that such a graph is regular of degree d, has order $n=2d^2-2$, has girth $g=6$ and diameter $D=4$. We start by giving a sketch of the argument that $D=4$. Consider two distinct lines $L_1=(1,a_1,b_1)$ and $L_2=(1,a_2,b_2)$. They are adjacent to the same point $P=(1,x,y)$ if and only if $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -1 \\ -1 \end{pmatrix}$. If this system has a solution, then the distance between these two lines is 2. In the opposite case, we have that $(a_2,b_2)=c\,(a_1,b_1)$ for some $c\in F$. In this case, pick a point P perpendicular to L_1. This point has $d-1$ other lines that are perpendicular to it. Pick one such line, call it $L_3=(1,a_3,b_3)$. Note that $(a_3,b_3)\neq c\,(a_2,b_2)$. for any $c\in F$. But then $\mathrm{dist}(L_2,L_3)=2=\mathrm{dist}(L_3,L_1)$ so that $\mathrm{dist}(L_1,L_2)=4$. Similar argument shows that $\mathrm{dist}(L,P)\leq 3$ for any line L and point P.

Conjectures and Open Questions

Conjectures:

- D or g-maximal graph of size n has the highest AC among <u>all</u> graphs of size n.

- A diameter-maximal graph of odd diameter D must have girth $g = D + 1$.
- A diameter-maximal graph of even diameter D must have a girth of either $g = D, D + 1$ or $D + 2$.

Open Questions:

- Find $d=3$, $D=10$ maximal graph.
- Do D-maximal graphs exist for any D?
- Are all Moore graphs girth-maximal?
- Find D-maximal graphs with $D=4$ and p <u>not</u> a prime power [$d=6$ exists]
- Find D-maximal family with $D=5$

Reference:

Exoo G, Kolokolnikov T, Janssen J, and Salamon T, *Attainable bounds for algebraic connectivity and maximally-connected regular graphs*, J Graph Theory. 2024