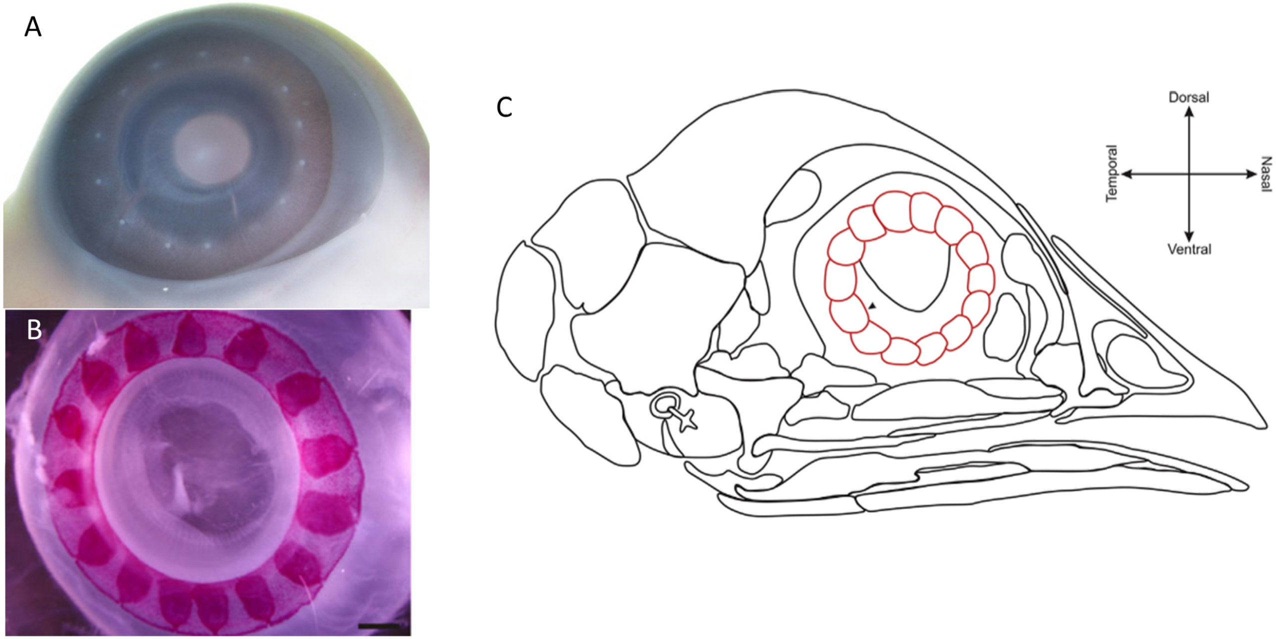
Stripe patterns for GM model in thin domains



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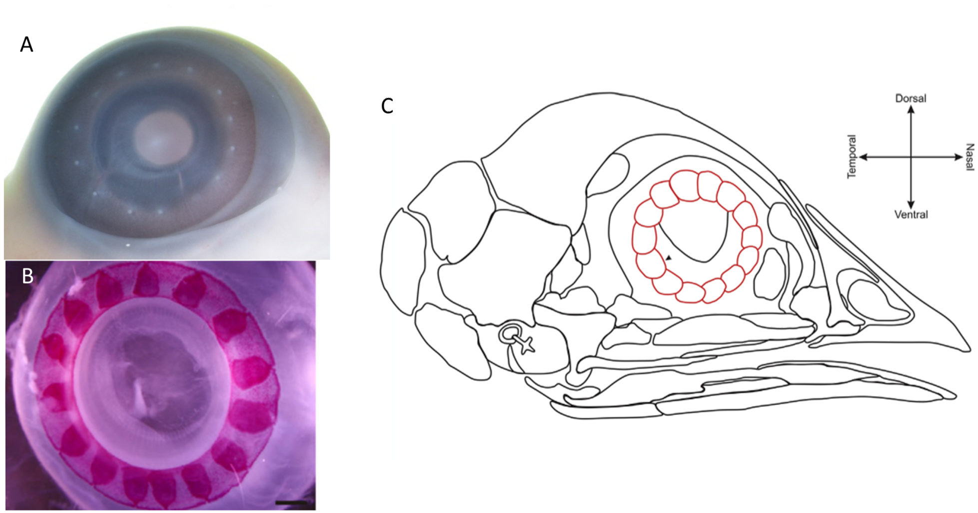
Joint work with Leila Mohammedi, David Iron and Tamara A. Franz-Odendaal

Motivation



Cnjunctival placodes within the embryonic eye of birds [Duench, Franz-Odendaal 2012; Drake-Jourdeuil-Franz-Odendaal 2024, Franz-Odendaal 2008]. These placodes are transient structures that induce underlying bones within the sclera of the eye. Comparative morphological studies by Franz-Odendaal and coworkers show that there is a specific conserved and universal sequence to the appearance of the placodes; experimental evidence to-date points to a Turing mechanism of pattern formation.

Thin domains:

GM model:

$$a_t = \varepsilon^2 \Delta a - a + a^p / h^q, \tag{1}$$

$$0 = d^2 \Delta h - h + \frac{1}{\varepsilon} a^m / h^s \tag{2}$$

Standard “quasi-1D” reduction:

$$\Delta \to \partial_{xx} + \frac{A'(x)}{A(x)} \partial_x. \tag{3}$$

GM model becomes:

$$a_t = \varepsilon^2 \left(a_{xx} + \alpha(x) a_x \right) - a + a^p / h^q, \tag{4}$$

$$0 = d^2 \left(h_{xx} + \alpha(x) h_x \right) - h + \frac{1}{\varepsilon} a^m / h^s \tag{5}$$

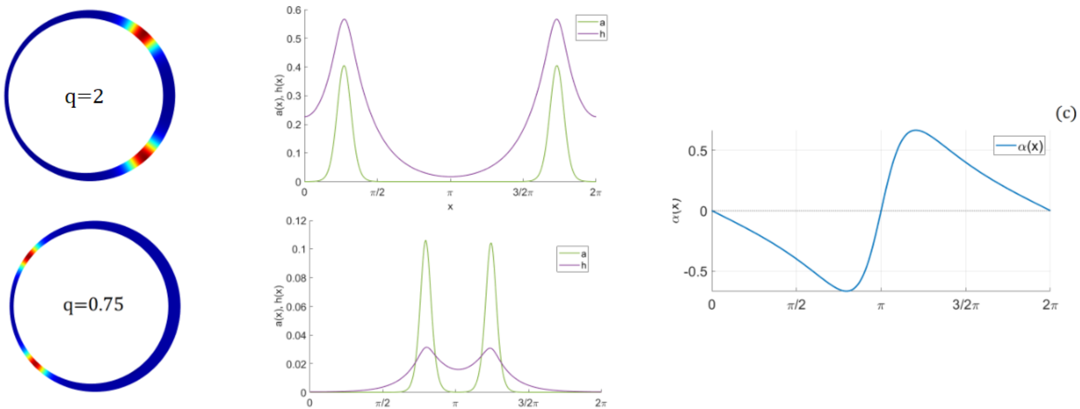
where $\alpha = A'/A,\ A$ is the cross-sectional area, x denotes the arclength coordinate along the boundary of the channel, ε is small.

Example:

Consider area between two circles. Outer circle is unit disk; inner circle off-center such that minimum and maximum distance from the outer boundary is $d_{\min}$ and $d_{\max}$, respectively, with $d_{\min} \le d_{\max} \le O(\varepsilon) \ll 1$. Then

$$\alpha(x) \sim \frac{1}{2} \frac{-\left(\kappa^2 - 1\right) \sin x}{1 + \kappa^2 + \left(\kappa^2 - 1\right) \cos x}; \quad \kappa = d_{\max}/d_{\min} \tag{6}$$

For concreteness we will take $\kappa = 3,\ p = 2,\ m = 2, s = 0$. Examples of equilibria:



p=2, m=2, s=0, with q as shown

Reduced system

In the limit $\varepsilon \ll d$, a quasi-equilibrium solution to (4) consisting of N stripes has the form

$$a(x,t) \sim \sum V_j^{q/(p-1)} w\left(\frac{x - x_j(\varepsilon^2 t)}{\varepsilon}\right), \tag{7}$$

$$H(x,t) \sim b_m \sum_j V_j^{\gamma} G\left(x, x_j\right) \tag{8}$$

where $w(y)$ is a standard ground state solution,

$$w_{yy} - w + w^p = 0; \quad w'(0) = 0; \quad w\left(y\right) \to 0 \text{ as } y \to \pm\infty, \tag{9}$$

$$b_m = \int_{-\infty}^{\infty} w^m(y) dy, \tag{10}$$

$G(x,\xi)$ is the Green's function satisfying

$$d^2\left(G_{xx} + \alpha(x) G_x\right) - G = -\delta(x - \xi), \tag{11}$$

and $x_j(t\varepsilon^2)$ is the time-dependent location of j-th stripe which is coupled to weights V_j and solves the following differential-algebraic system:

$$x_k' = -\alpha(x_k) - 2b_m \frac{q}{p-1} \frac{1}{V_k} \sum_j V_j^\gamma G_x(x_k, x_j) \tag{12}$$

$$V_k = b_m \sum_j V_j^\gamma G(x_k, x_j); \tag{13}$$

$$\gamma = \frac{qm}{p-1} - s; \tag{14}$$

where $G_x(x_k, x_k) = \frac{1}{2}\left(G_x(x_k^+, x_k) + G_x(x_k^-, x_k)\right)$.

Green's function WKB theory in the limit $d \ll 1$

$$d^2\left(G_{xx} + a(x)G_x\right) - G = -\delta(x-\xi); \qquad d \ll 1.$$

We use the standard WKB ansatz:

$$G \sim Y(x)e^{\frac{\phi(x)}{d}}$$

to obtain

$$\phi'^2 = 1; \ \frac{1}{Y}\left(Y^2\phi'\right)' + Y\phi'\beta = 0.$$

Since we require decay at infinity, we take $\phi' = -\operatorname{sign}(x-\xi)$ and equation for Y yields

$$Y = C\exp\left(-\frac{1}{2}\int_{\xi}^{x}\alpha(s)\,ds\right).$$

Applying the jump condition $d^2G_x|_{\xi^-}^{\xi^+} = -1$; then yields

$$G(x,\xi) \sim \frac{1}{2d}\exp\left(-\frac{1}{2}\int_{\xi}^{x}\alpha(s)\,ds\right)\exp\left(-\frac{|x-\xi|}{d}\right).$$

In particular we have:

$$G(\xi,\xi) \sim \frac{1}{2d} \tag{15}$$

$$\frac{G_x(\xi^+,\xi) + G_x(\xi^-,\xi)}{2} \sim -\frac{\alpha\,(\xi)}{4d}; \tag{16}$$

Single spike

$$x_0' = -\alpha(x_0) - 2\frac{q}{p-1}\frac{G_x(x_0,x_0)}{G(x_0,x_0)}. \tag{17}$$

- Large d limit ("shadow system"):

$$x_0' \sim -\alpha(x_0), \quad d \gg 1. \tag{18}$$

  - Equilibrium location x_0 then correspond to zeros of $\alpha(x) = A'(x)/A(x)$, with ***stable equilibrium corresponding to the minimum*** of cross-sectional area $A(x)$.

- Small d : Using WKB for Green's function we obtain

$$x_0' \sim \left(-1 + \frac{q}{p-1}\right)\alpha(x_0), \quad d \ll 1. \tag{19}$$

  - Equilibria still at zero of $\alpha(x) = A'(x)/A(x)$, but their stability can ***switch*** depending on the sign of $q+1-p$: if $p > q+1$, stable equilibria correspond to the minimum of A, and unstable to the maximum of A; this is reversed when $p < q+1$.
  - ***Borderline case***: $q = p-1$. This includes the "standard choice" $p = 2, q = 1$. Then $x' \sim 0$ up to $O(\varepsilon^2)$. Higher-order expansion would be needed to capture the motion.

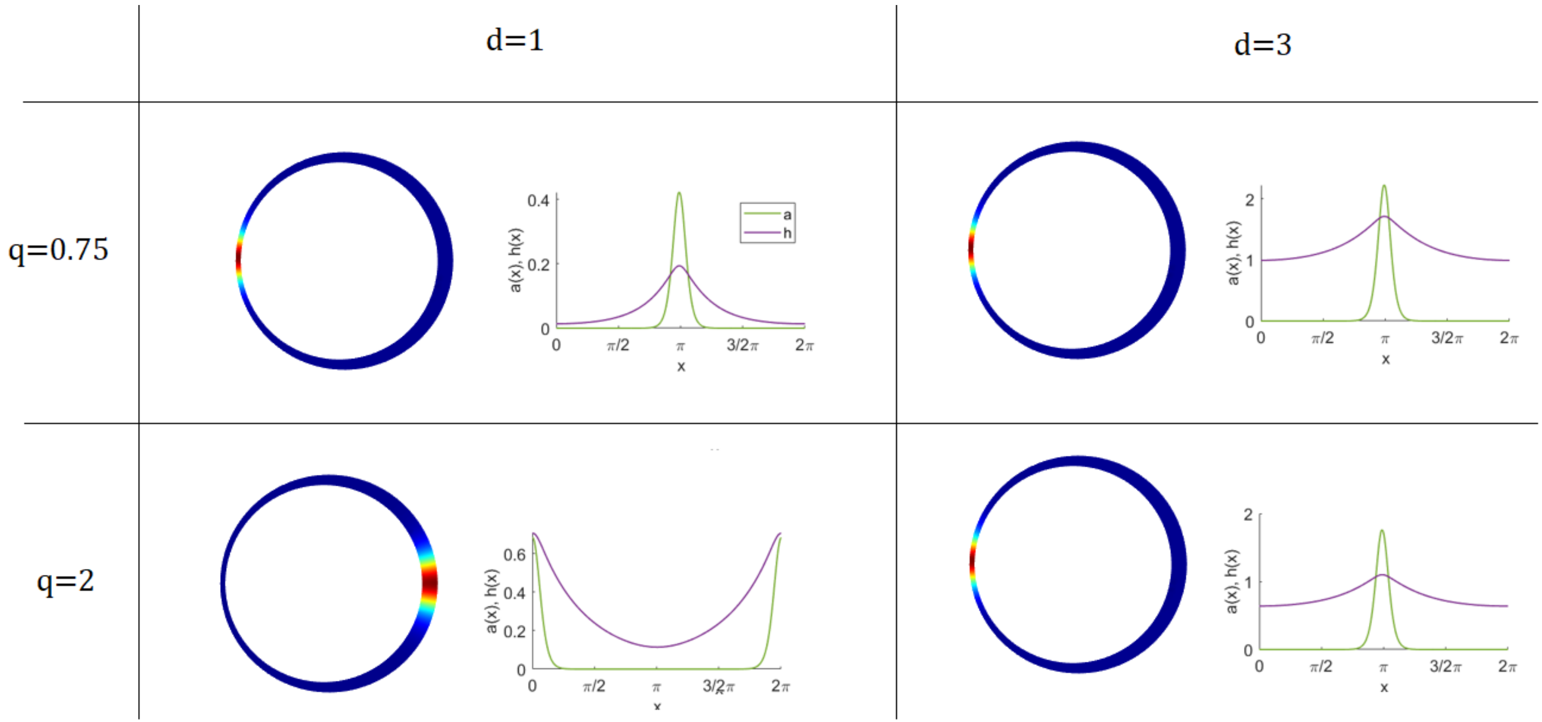


Figure 1: Stable equilibrium solutions of the full 2D system (**??**) consisting of a single spike inside an annular region of uneven thickness. The outer boundary is a unit circle. The inner boundary is a circle which is at a minimum distance of 0.05 and a maximum distance of 0.15 from the outer boundary. Full 2D time-dependent problem was solved using an initial condition consisting of a single spike at $x = \pi/2$. Snapshots show the solution after it converged to a (stable) equilibrium. Parameters d and q are as shown, while other parameters are $\varepsilon = 0.1, \;\; p, m, s = 2, 2, 0$. Note that the spike is expected to be stable at the thinnest part of the domain when d is large, regardless of q. When d is small, the spike is stable at the thinnest part if $q < p - 1$, but is stable at the thickest part

Two spikes

- **Two symmetric stripes:** $x_1 = r,\ \ x_2 = -r$:

$$r' \sim -\alpha(r) - 2\frac{q}{p-1}\frac{G_x(r,r) + G_x(r,-r)}{G(r,r) + G(r,-r)}.$$

Assume that $\varepsilon \ll r \ll 1$, WKB yields

$$r' \sim \frac{\frac{2q}{p-1}\frac{1}{d}\exp\left(-\frac{2r}{d}\right)}{1+\exp\left(-\frac{2r}{d}\right)} + \alpha(r)\left(-1+\frac{q}{p-1}\right). \tag{20}$$

Boundary case: if $q = p-1$, then $r \to \infty$ regardless of α; no “cluster”

Otherwise, Expand near $r \sim 0$, we get

$$r' \sim \frac{\frac{2q}{p-1}\frac{1}{d}\exp\left(-\frac{2r}{d}\right)}{1+\exp\left(-\frac{2r}{d}\right)} + r\alpha'(0)\left(-1+\frac{q}{p-1}\right). \tag{21}$$

More generally,

Suppose that $\alpha(x)$ has a root at $x = x_0$ and suppose that $\alpha'(x_0)(q-p-1) < 0$. Then there is a two-spike steady state with spike positions at $x \sim x_0 \pm r$, where $r = O(d\log d^{-1})$ asymptotes to the small solution of

$$e^{2r/d} \sim \frac{1}{r}\frac{1}{d}\frac{2q}{\alpha'(x_0)(p-1-q)}. \tag{22}$$

No such solution exists if $q = p-1$ or $\alpha'(x_0)(q-p-1) > 0$

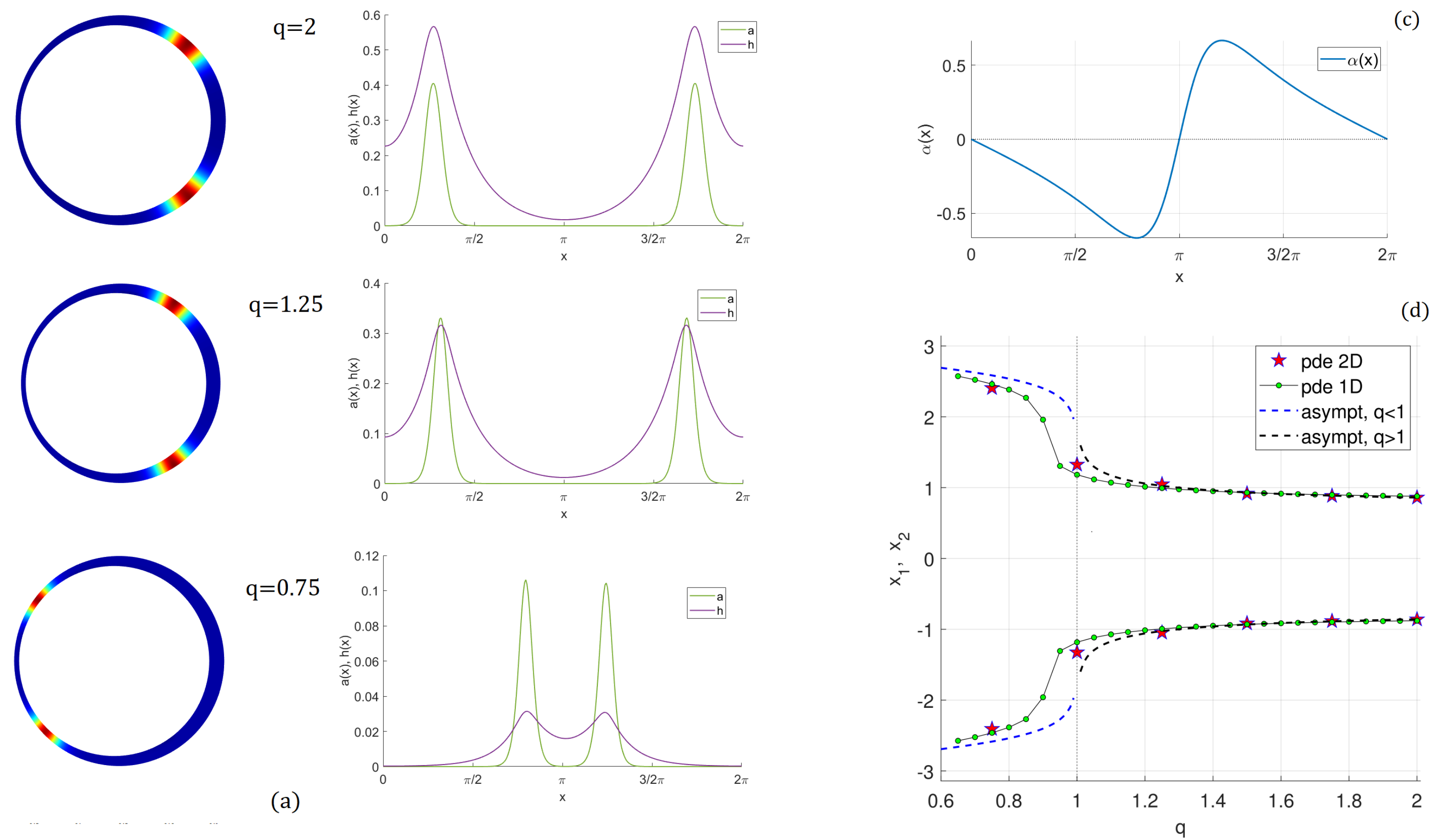


Figure 2: Two spikes inside an annular region of uneven thickness. The outer boundary is a unit circle. The inner boundary is the circle through points $-1 + 0.05,\ \ 1 - 0.15$ (i.e. $\kappa = 3$ in (**??**)). The first column shows the equilibrium state for the full 2D system (**??**). Second column shows the corresponding equilibrium in the 1D system (4), approximating the solution along the boundary of the domain. (a-c) Parameter values are $\varepsilon = 0.07$, $D = 0.2$, $p, m, s = 2, 2, 0$, and with q as indicated. (c) The sketch of

Many stripes

- We derive the effective spike density using methods from [Kolokolnkov, Xie, 2020]
- Idea: in the limit d small, the Green's function interaction is exponentially weak. So each spike only feels a local neigbourhood of spikes.
- Sort spike positions in increasing order, $x_1 < x_2 < \ldots < x_N$. Assume typical inter-spike distance of $O(d)$. Define $u(x)$ such that $x_{j+1} - x_j \approx u(x_j)d$. Expand in Taylor series. End-result: for equilibrium cluster,

$$u_x = \alpha(x)\frac{\left(\left(\frac{p-1}{q} - 1\right)\frac{\sinh u}{u} + 1\right)\sinh(u)}{\cosh(u) - \frac{\gamma+1}{\gamma-1}}, \quad \gamma = \frac{qm}{p-1} - s. \tag{23}$$

  This is a separable ODE and determines the effective inter-spike distance ud in the limit of large N.

- The density distribution is given by $\rho(x) = 1/u(x)$, where $u(x)$ satisfies (23) subject to an integral constraint

$$\int_D \rho(x)dx = Nd. \tag{24}$$

  where $D = (a, b)$ is the support of $\rho(x)$ (i.e. u blows up $x = a, b$).

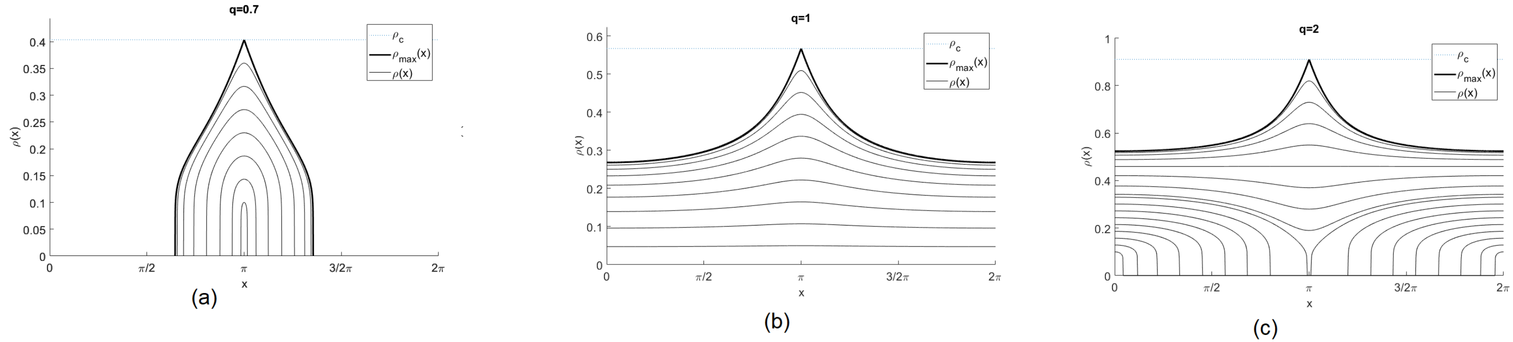
- Note that (23) has a singularity when $\cosh(u) = \frac{\gamma+1}{\gamma-1}$. Competition instability occurs u is below this threshold [K-X 2020]. Let

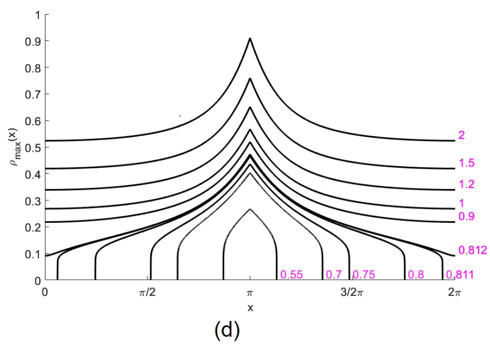
$$u_c := \operatorname{arccosh}\left(\frac{\gamma+1}{\gamma-1}\right), \quad \rho_c := 1/u_c. \tag{25}$$

  This critical threshold agrees with known results about competition instability [Iron-Ward-Wei, 2001]. In particular, the inter-spike distance must be larger than $u_c d$.

- We call the density $\rho(x) = 1/u(x)$ "***admissible***" if $\rho(x) \leq \rho_c$ for all x, where ρ_c is given by (25). We call $\rho(x)$ ***maximal admissible density***, denoted it by $\rho_{\max}(x)$, if $\rho_{\max}(x) \geq \rho(x)$ for any admissible density $\rho(x)$.

- Typically, Turing instability results in more spikes than $\rho_{\max}$ can support due to large unstable band ($\varepsilon \ll d$). Therefore it is followed by coarsening process until $\rho \sim \rho_{\max}$ (at the steady state)

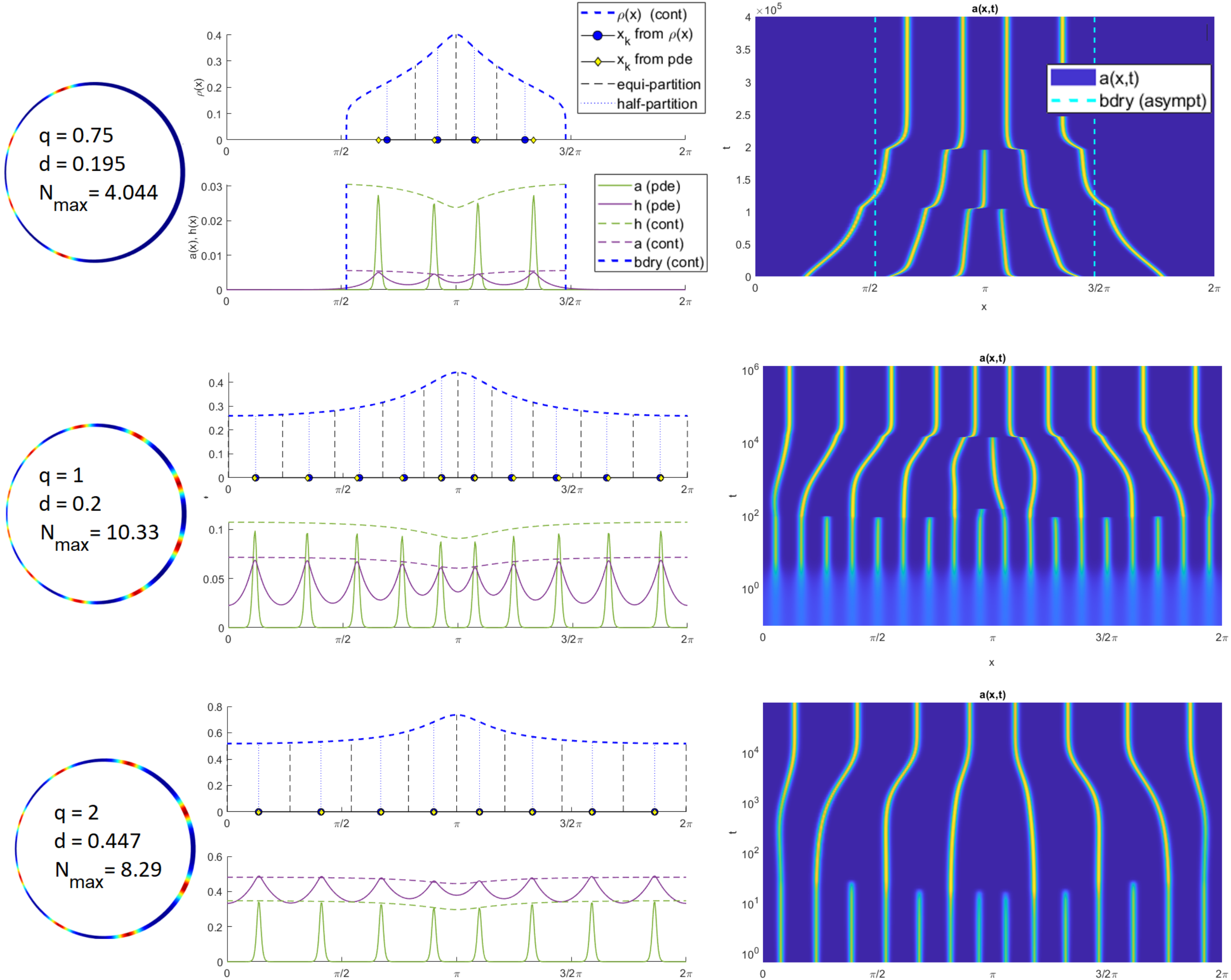




q	$R_{\max}$	$d\ N_{\max}$
0.6000	0.6131	0.2898
0.7000	1.1105	0.5765
0.8000	2.3701	1.1556
0.8500	π	1.6106
0.9000	π	1.7894
0.9500	π	1.9367
1.0000	π	2.0671
1.2500	π	2.5942
1.5000	π	3.0159
2.0000	π	3.7079

(e)

Figure 3: (a-c): Solutions of ODE (23) for the density $\rho(x) = 1/u(x)$ corresponding to different admissable initial conditions (for which $\rho(x) < \rho_c$) with q as indicated, and with $\alpha(x)$ as in (**??**) and $p = 2, m = 2, s = 0$. (d) Plot of $\rho_{\max}(x)$ as a function of q, other parameters are as in (a-c). (e) $N_{\max}d$ and cluster size $R_{\max}$ as a function of q, see text.



q = 0.75
d = 0.195
N max = 4.044
ρ(x) (cont)
x k from ρ(x)
x k from pde
equi-partition
half-partition
a (pde)
h (pde)
h (cont)
a (cont)
bdry (cont)
a(x), h(x)
ρ(x)
a(x,t)
bdry (asympt)
x
t
q = 1
d = 0.2
N max = 10.33
q = 2
d = 0.447
N max = 8.29

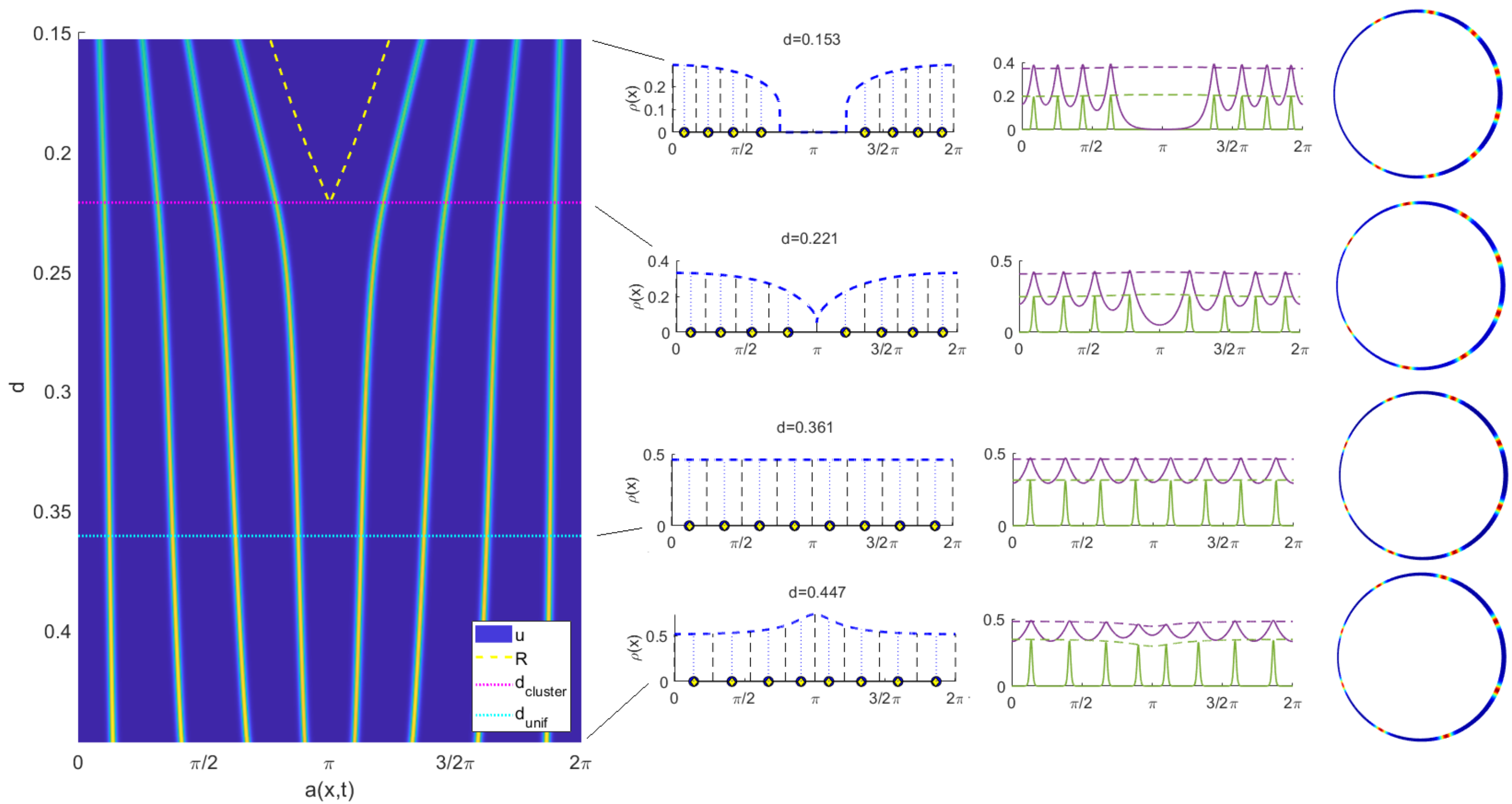


Figure 5: Simulations of (**??**) starting with random initial conditions, with $\varepsilon = 0.018,\ p = q = m = 2, s = 0$ and with slowly decreasing d according to $d = 0.447 + 10^{-7}(0.153 - 0.447)t$, $t = 0 \ldots 10^7$. Snapshots show direct comparison with continuum density for d as indicated as well as the corresponding steady state. Note that the spike distribution is uniform when $d = 0.361$ (see text); and a cluster starts to form around $d = 0.221$, in full agreement with the theory.

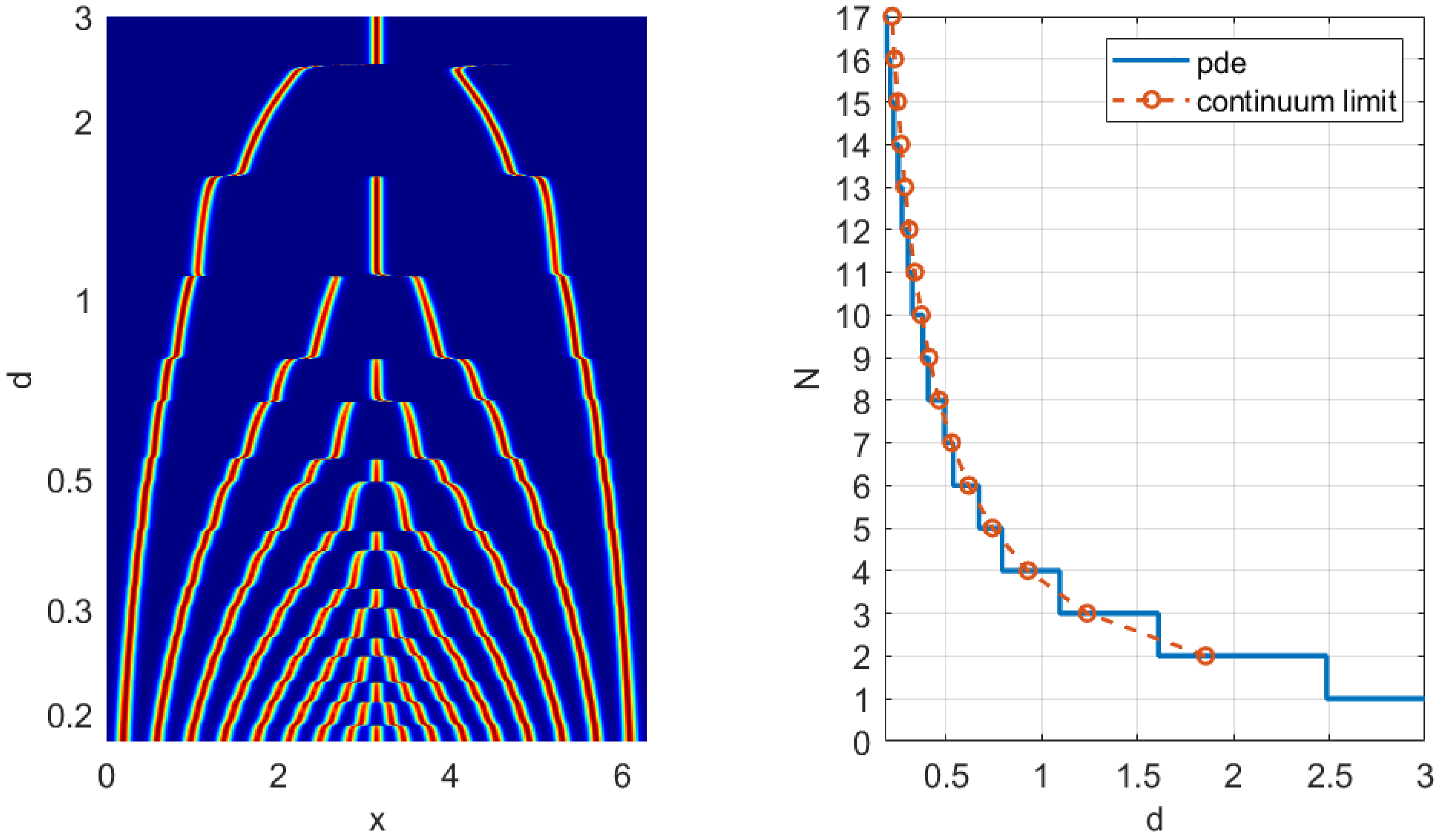


Figure 6: Simulations of (**??**) starting with random initial conditions, with $\alpha(x)$ as in (**??**), $\varepsilon = 0.03,\ p = q = m = 2, s = 0$ and with slowly increasing d according to $d = 0.16 + (3 - 0.0.16)10^{-6}t$, $t = 0 \ldots 10^{6}$. Left: Full solution to the PDE. Right: N, number of spikes, as a function of d : comparison between the full PDE and the continuum theory (see text)

Discussion

- We have described stripe evolution and their equilibrium distributions on thin channels for the GM model.
- We found clusters can concentrate either near ***thinnest*** or ***thickest*** part of the channel, depending on parameters. This is in contrast to interface-minimizing systems such as Allen-Cahn, where the dynamics push the interface to minimize its length and therefore be located at the ***thinnest*** part of the channel [Kohn-Sternberg,1989; Chen, 1992; Iron-Kolokolnkov-Rumsay-Wei, 2009].
- When the domain is too thick ($\geq O(\varepsilon)$), stripes break up into spots [Doelman and van der Ploeg, 2002]. ***Open question:*** analyse motion of ***spots*** (rather than stripes) inside a thin channel. Study co-existing patterns – as in the picture.
- Reference: Leila Mohammadi, Theodore Kolokolnikov, David Iron, and Tamara A. Franz-Odendaal, “Stripe patterns for Gierer-Meinhard model in thin domains”, to appear, Physica D.

Thank you!

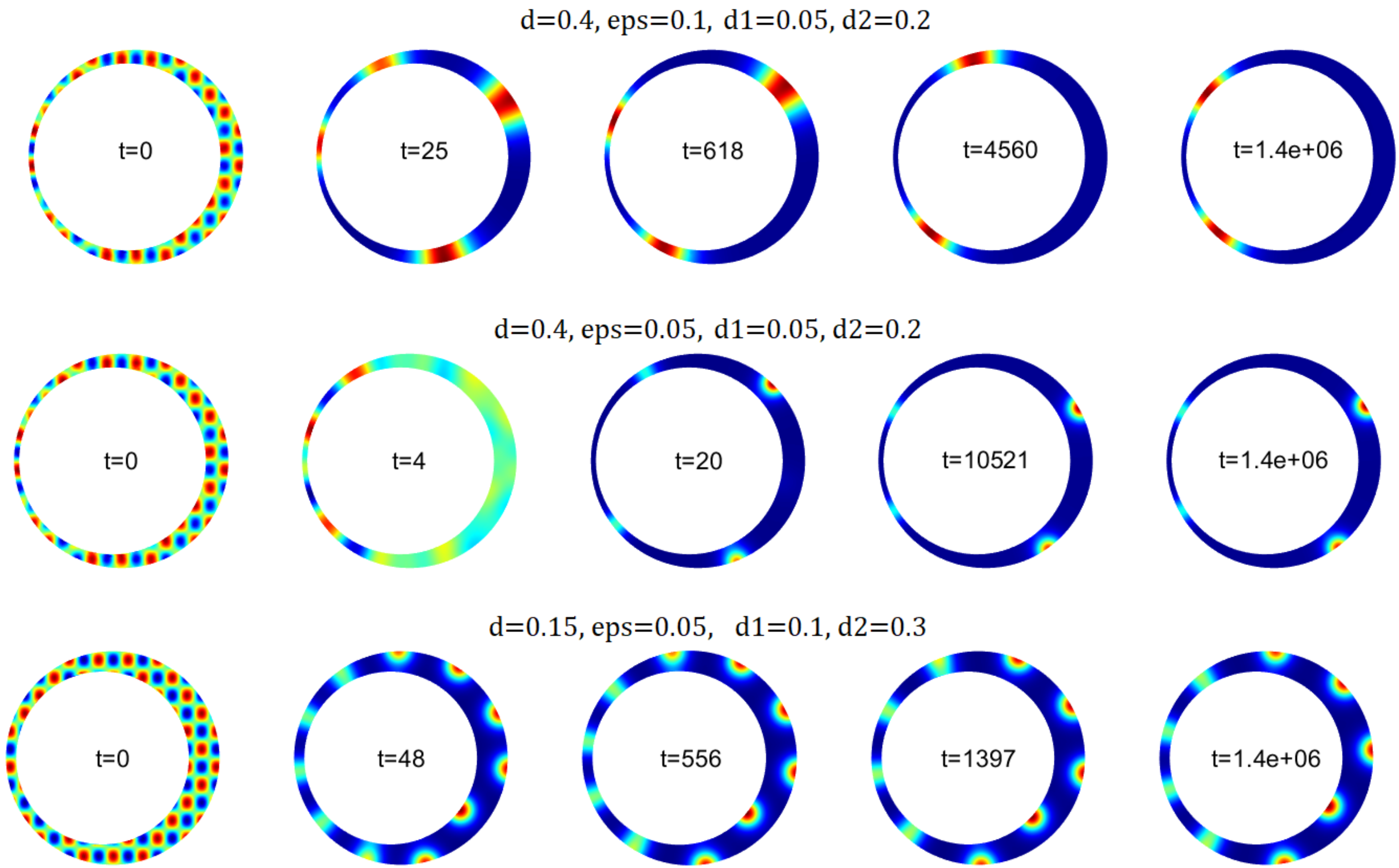


Figure 7: Coexistence of stripes and spikes. Full simulation of (**??**) inside a channel whose outer boundary is a unit circle and whose inner boundary is a smaller circle with d_1 and d_2 is the minimum and maximum thickness of the channel. Parameters are $(p, q, m, s) = (2, 0.75, 2, 0)$ with other parameters as indicated. First row: for $\varepsilon = 0.1$,